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Ilya Molchanov

Theory of Random Sets

Second Edition

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Ilya Molchanov
Institute of Mathematical Statistics
and Actuarial Science
University of Bern
Bern, Switzerland

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To my mother

Preface

Some History

The study of random geometrical objects goes back to the famous Buffon needle problem. Similar to the ideas of Geometric Probability, which can be traced back to the very origins of probability, the concept of a random set was mentioned for the first time together with the mathematical foundations of Probability Theory. A.N. Kolmogorov [493, p. 46] wrote in 1933 (translated from German):

> Let G be a measurable region of the plane whose shape depends on chance; in other words, let us assign to every elementary event ξ of a field of probability a definite measurable plane region G. We shall denote by J the area of the region G and by $\mathbf{P}(x, y)$ the probability that the point (x, y) belongs to the region G. Then
>
> $$\mathbf{E}(J) = \iint \mathbf{P}(x, y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

One might observe that this is a formulation of Robbins' theorem and $\mathbf{P}(x, y)$ is the coverage function of the random set G.

Further progress in the theory of random sets relied on developments in the following areas:

- studies of random elements in general topological spaces, in groups and semi-groups, see, e.g., Grenander [326];
- the general theory of stochastic processes, see Dellacherie [220], and the theory of capacities, see Choquet [172];
- set-valued analysis and multifunctions, see Castaing and Valadier [158];
- advances in image analysis and microscopy that required a satisfactory mathematical theory of distributions for binary images (or random sets), see Serra [790].

The mathematical theory of random sets can be traced back to Matheron [581] and Kendall [454]. The principal new feature is that random sets may have different shapes and the development of this idea is crucial in the study of random sets.

G. Matheron formulated the very definition of a random closed set and developed the relevant probabilistic and geometric techniques. D.G. Kendall's seminal paper [454] on random sets already contained the first steps into many further directions such as lattices, weak convergence, spectral representation, infinite divisibility. Most of these aspects were elaborated later on in connection with relevant ideas in pure mathematics and classical probability theory. This has made many of the concepts and the notation used in [454] obsolete, so we will follow instead the modern terminology that fits better into the system developed by G. Matheron; most of his notation was taken as the basis for this monograph.

The relationship between random sets and convex geometry later on has been thoroughly explored within the stochastic geometry literature, mostly in the stationary setting, see, e.g., Schneider and Weil [780]. Within stochastic geometry, random sets represent one type of object along with point processes and random tessellations, see Chiu, Stoyan, Kendall and Mecke [169]. The mathematical morphology part of G. Matheron's book gave rise to numerous applications in image processing (Dougherty [239] and Serra [790]) and abstract studies of operations with sets, often in the framework of lattice theory (Heijmans [355]).

Since 1975, when G. Matheron's book [581] was published, the theory of random sets has enjoyed substantial developments concerning

- relationships to the theories of semigroups and continuous lattices;
- properties of capacities;
- limit theorems for Minkowski sums based upon techniques from probabilities in Banach spaces;
- limit theorems for unions of random sets in relation to the theory of extreme values;
- stochastic optimisation ideas in relation to random sets that appear as epigraphs of random functions;
- properties of level sets and excursions of stochastic processes.

These developments constitute the core of this book, which aims to cast the theory of random sets into the conventional probabilistic framework that involves distributional properties, limit theorems and related analytical tools.

Central Topics of the Book

This book concentrates on several basic concepts in the theory of random sets. The first is the *capacity functional* that determines the distribution of a random closed set in a locally compact Hausdorff separable space. Unlike probability measures, the capacity functional is *non-additive*. The studies of non-additive set functions are abundant, especially, in view of game theory applications to describe the gain attained by a coalition of players, in statistics as belief functions in order to model situations where the underlying probability measure is uncertain, and in mathematical finance, where non-additive set functions are essential to assess risk.

The capacity functional can be used to characterise the weak convergence of random sets and some properties of their distributions. In particular, this concerns unions of random closed sets, where the regular variation property of the capacity functional is of primary importance. However, the capacity functional does not help to deal with a number of other issues, for instance to define the expectation of a random closed set.

Here the leading role is taken over by the concept of a *selection*, which is a (single-valued) random element that almost surely belongs to a random set. In this framework, it is convenient to view a random closed set as a multifunction (or set-valued function) on a probability space and use the well-developed machinery of set-valued analysis, see, e.g., Hu and Papageorgiou [402]. By taking expectations of integrable selections, one defines the *selection expectation* of a random closed set. The selection expectation of a random set defined on a non-atomic probability space is always convex and can be alternatively defined as the convex set whose *support function* equals the expected support function of a random set. The *Minkowski sum* of random sets is introduced as the set of sums of all their points (or all their selections) and can be equivalently defined using the arithmetic sum of the support functions. Therefore, limit theorems for Minkowski sums of random sets can be derived from the existing results for random elements in functional spaces. These tools make it possible to explore *set-valued martingales*.

Important examples of random closed sets appear as *epigraphs* of random lower semicontinuous functions. Viewing the epigraphs as random closed sets makes it possible to obtain results for lower semicontinuous functions under the weakest possible conditions. In particular, this concerns the convergence of minimum values and minimisers, which is a subject of stochastic optimisation theory.

It is possible to consider the family of closed sets as both a *semigroup* and a *lattice*. Therefore, the results on lattice- or semigroup-valued random elements are very useful in the theory of random sets.

Plan

Since the concept of a set is central for mathematics, the book is highly interdisciplinary and relies on tools from a number of mathematical theories and concepts: capacities, convex geometry, set-valued analysis, topology, harmonic analysis on semigroups, continuous lattices, non-additive measures and upper/lower probabilities, limit theorems in Banach spaces, the general theory of stochastic processes, extreme values, stochastic optimisation, point processes and random measures.

The book starts with the definition of a random closed set. The space $\mathbb{E}$ which random sets belong to is very often assumed to be locally compact Hausdorff with a countable base. The Euclidean space $\mathbb{R}^d$ is a generic example. Often we switch to the more general case of $\mathbb{E}$ being a Polish space or Banach space (if a linear structure is essential). It is convenient to work with random *closed* sets, which is the typical setting in this book, although in some places we mention random open sets and random Borel sets. Choquet's theorem concerning the existence of random set distributions is proved and relationships with set-valued analysis (or multifunctions) and lattices are explained. The rest of Chap. 1 relies on the concept of the capacity functional. It highlights relationships between capacity functionals and properties of random sets, develops some analytic theory, convergence concepts, applications to point processes and random capacities and finally surveys various interpretations for capacities that stem from game theory, imprecise probabilities and robust statistics. Special attention is devoted to the case of random convex compact sets (or convex bodies if the carrier space is Euclidean).

Chapter 2 concerns expectation concepts for random closed sets. The main part is devoted to the selection (or Aumann) expectation based on the idea of an integrable selection. Chapter 3 continues this topic by dealing with Minkowski sums of random sets. The dual representation of the selection expectation—as the set of expectations of all selections and as the expectation of the support function—makes it possible to refer to limit theorems in Banach spaces in order to derive the corresponding results for random closed sets.

The study of unions for random sets is closely related to extremes of random variables and further generalisations for pointwise extremes of stochastic processes. Chapter 4 describes the main results for the unions of random sets and explains the background ideas that are related to the studies of lattice-valued random elements and regular variation on abstract spaces.

Chapter 5 is devoted to links between random sets and stochastic processes. This concerns set-valued processes that develop in time, in particular, set-valued martingales. Furthermore, this relates to random sets interpretations of conventional stochastic processes, where random sets appear as graphs, level sets or epigraphs (hypographs). Several areas related to random sets and stochastic processes are only mentioned in brief, for instance, the theory of set-indexed processes, where random sets appear as stopping times (or stopping sets), excursions of random fields, and potential theory for Markov processes that provides further examples of capacities related to hitting times and paths of stochastic processes.

The Appendices summarise the necessary mathematical background; it stems from various parts of mathematics and is normally scattered between various texts.

Second Edition

The period between the first and second editions witnessed the appearance of several books on stochastic geometry and random sets authored by Nguyen [651], Schneider and Weil [780], Chiu, Stoyan, Kendall and Mecke [169], on random measures by Kallenberg [444], Poisson point processes by Last and Penrose [526], and on non-additive measures by Grabisch [321] and Cuzzolin [196].

The second edition of this book includes new material in the following directions:

- unbounded and possibly non-closed random sets motivated by applications in mathematical finance in order to describe set-valued portfolios;
- selections of random sets, motivated by the use of random sets to describe partially identified models in econometrics;
- random closed (compact) sets in Polish spaces;
- regular variation and stability of random elements in abstract spaces;
- sublinear and superlinear expectations of random sets, motivated by applications to risk assessment;
- results on transformations of capacities and rearrangement invariant random closed sets;
- relationships between random sets and multivariate probability theory, mostly using the concept of zonoids, connections to stable laws and multivariate extremes, series representations of stable laws;
- for Minkowski sums, a new Marcinkiewicz–Zygmund strong law of large numbers is proved, and results on large deviations for sums of heavy-tailed random sets are mentioned;
- continuous time set-valued processed are discussed in depth, including the separability concept, graphical convergence, and uniform laws of large numbers.

In the second edition, the locally compact and infinite-dimensional settings are more clearly identified, and it has been made clearer which results hold for unbounded random closed sets and for non-closed random sets. The measure-theoretic proof of Choquet's theorem has been corrected and a new proof following the idea of relative compactness has been added. The characterisation of selections is now presented with a full proof. The presentation of the union scheme has been restructured by the systematic use of regular variation in abstract spaces, and both the union- and sum-stability concepts are brought in relation to series representations. The presentation of the selection expectation is accompanied by the discussion and the proof of the Aumann identity; full proofs of the properties of conditional expectations are now included, and the generalised selection expectation is introduced. Results on extremal processes are brought in relation to the recent work on capacities; a new streamlined proof for the properties of continuous choice processes is now presented.

The second edition gave a chance to correct numerous misprints, occasional mistakes and misinterpretations. While the chapter structure remained the same, the presented material has undergone lots of substantial changes to the extent that this edition may well be considered a completely rewritten text. It includes also numerous references to papers on random sets and their applications published since 2005.

Conventions

The numbering in the second edition follows a three-digit pattern, where the first digit is the chapter number followed by section. When referring to the Appendices, the first two digits are replaced by a letter that designates the particular appendix. The statements in theorems and propositions are mostly designated by Roman numerals, while the conditions usually follow the Arabic numeration.

Although the main concepts in this book are used throughout the whole text, it is anticipated that the reader will be able to read the book from the middle. The concepts are often restated, definitions recalled, and the notational system is set to be as consistent as possible, taking into account various conventions within a number of mathematical areas that build up this book.

Future supporting information for this book (e.g., the eventual list of misprints or comments to open problems) will be available through Springer's WEB site or from the author's personal page, which can easily be found with search engines.

Acknowledgements

G. Matheron's book *Random Sets and Integral Geometry* [581] accompanied me throughout my whole life in mathematics since 1981 where I first saw its Russian translation (published in 1978). Then I became fascinated in this cocktail of techniques from topology, convex geometry and probability theory that essentially makes up the theory of random sets.

This book project (over the two editions) has spanned my work and life in five different countries: Germany, the Netherlands, Scotland, Spain and Switzerland. I would like to thank the people of all these and many other countries who supported me at various stages of my work and from whom I had a chance to learn. In particular, I would like to thank Dietrich Stoyan who, a while ago, encouraged me to start writing this book, and my colleagues in Bern for a wonderful working and living environment. The Swiss National Science Foundation (SNF) has been supporting my research work for many years.

A lot of motivation for the second edition came from economical and financial applications. I am grateful to Francesca Molinari for bringing the theory of random sets to econometrics and for explaining to me relevant problems.

I am grateful to the creators of the XEmacs software which was absolutely indispensable during my work on this large LaTeX project and to the staff of Springer who helped me to complete this work.

Finally, I would like to thank my family and close friends for being always ready to help, whatever happens.

Bern, Switzerland — Ilya Molchanov
May 2017

Contents

Chapter 1
Random Closed Sets and Capacity Functionals

1.1 Distributions of Random Sets

1.1.1 Set-Valued Random Elements

Measurability Definition

As the name suggests, a random set is an object with values being sets, so that the corresponding record space is the space of subsets of a given carrier space. At this stage, a mere definition of a general random element like a random set presents little difficulty as soon as a σ-algebra on the record space is specified.

Because the family of *all* sets is rather rich, it is usual to consider random sets with some extra conditions on their possible values, e.g., closed, open, or convex. In order to include the case of random singletons (which are closed in topological spaces satisfying rather mild requirements), it is common to consider random *closed* sets. The family of closed subsets of a topological space $\mathbb{E}$ is denoted by $\mathcal{F}$, while $\mathcal{K}$ and $\mathcal{G}$ denote, respectively, the family of all compact and open subsets of $\mathbb{E}$. It is often assumed that $\mathbb{E}$ is a *locally compact Hausdorff second countable* topological space (LCHS space). The Euclidean space $\mathbb{R}^d$ is a generic example of such a space $\mathbb{E}$.

Let us fix a complete probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ which will be used throughout to define random elements. It is natural to call an $\mathcal{F}$-valued random element a random closed set. However, one should be more specific about measurability issues, in other words, when defining a random element it is necessary to specify which information is available in terms of the observable events from the σ-algebra $\mathfrak{A}$ in Ω. It is essential to ensure that the measurability requirement is restrictive enough, so that all functionals of interest become random variables. At the same time, the measurability condition must not be too strict in order to include as many random elements as possible. The following definition describes a rather flexible and useful concept of a random closed set.

I. Molchanov, *Theory of Random Sets*, Probability Theory and Stochastic Modelling 87, DOI 10.1007/978-1-4471-7349-6_1

Definition 1.1.1 (Definition of a random closed set) A map $X\colon \Omega \mapsto \mathcal{F}$ from a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ to the family of closed sets in an LCHS space $\mathbb{E}$ is called a *random closed set* if, for every compact set K in $\mathbb{E}$

$$\{\omega : X(\omega) \cap K \neq \emptyset\} \in \mathfrak{A}. \tag{1.1.1}$$

Although we will postpone considering of random closed sets in more general spaces until Sect. 1.3.1, we give here the definition of random closed sets in Polish spaces, which is equivalent to the above definition if the carrier space $\mathbb{E}$ is LCHS.

Definition 1.1.1′ A map $X\colon \Omega \mapsto \mathcal{F}$ from a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ to the family of closed sets in a Polish space $\mathbb{E}$ is called a *random closed set* if, for every open set G in $\mathbb{E}$

$$\{\omega : X(\omega) \cap G \neq \emptyset\} \in \mathfrak{A}. \tag{1.1.2}$$

Condition (1.1.1) means that observing X one can always say if X hits or misses any given compact set K. In more abstract language, (1.1.1) says that the map $X\colon \Omega \mapsto \mathcal{F}$ is measurable as a map between the underlying probability space and the space $\mathcal{F}$ equipped with the σ-algebra $\mathfrak{B}(\mathcal{F})$ generated by $\{F \in \mathcal{F} : F \cap K \neq \emptyset\}$ for K running through the family $\mathcal{K}$ of compact subsets of $\mathbb{E}$.

Denote the family of closed sets that hit any given $A \subset \mathbb{E}$ by

$$\mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \emptyset\},$$

so that $\mathcal{F}_K$ is the family of closed sets that hit $K \in \mathcal{K}$. Since the σ-algebra $\mathfrak{B}(\mathcal{F})$ is generated by $\mathcal{F}_K$ for all K from $\mathcal{K}$, this σ-algebra clearly contains the complements to $\mathcal{F}_K$. These complements are denoted by

$$\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\},$$

so that $\mathcal{F}^K$ is the family of closed sets missing K.

The topological assumptions on $\mathbb{E}$ are important in the following proposition, which establishes the equivalence of Definition 1.1.1 and Definition 1.1.1'. It confirms that $\mathfrak{B}(\mathcal{F})$ coincides with the Effros σ-algebra discussed in greater detail in Sect. 1.3.1 for the case of a general Polish space $\mathbb{E}$.

Proposition 1.1.2 *If $\mathbb{E}$ is LCHS, then the σ-algebra $\mathfrak{B}(\mathcal{F})$ is countably generated and coincides with the σ-algebra generated by $\mathcal{F}_G$ for G running through the family $\mathcal{G}$ of open subsets of $\mathbb{E}$*

Proof. By Proposition A.1, each $K \in \mathcal{K}$ can be approximated by a sequence of open sets $\{G_n, n \geq 1\}$, so that $G_n \downarrow K$, whence

$$\mathcal{F}_K = \bigcap_{n\geq 1} \mathcal{F}_{G_n}.$$

Furthermore, for every G from the family $\mathcal{G}$ of open sets,

$$\mathfrak{F}_G = \{F \in \mathfrak{F} : F \cap G \neq \emptyset\} = \bigcup_n \mathfrak{F}_{K_n} \in \mathfrak{B}(\mathfrak{F}),$$

where $\{K_n, n \geq 1\}$ is a sequence of compact sets such that $K_n \uparrow G$ (here the local compactness of $\mathbb{E}$ is essential, see Proposition A.1). Taking relatively compact sets from a countable base of the topology on $\mathbb{E}$ confirms that $\mathfrak{B}(\mathfrak{F})$ is countably generated. □

Corollary 1.1.3 *Let $\mathbb{E}$ be LCHS. A map $X: \Omega \mapsto \mathfrak{F}$ is a random closed set if and only if $\{X \cap K \neq \emptyset\} \in \mathfrak{A}$ for all $K \in \mathfrak{M}$, where $\mathfrak{M}$ is any family of compact sets, such that any open set appears as a countable union of sets from $\mathfrak{M}$.*

Proof. Only sufficiency requires a proof. Since each open set G is obtained as the union of compact sets $K_n \in \mathfrak{M}$, $n \geq 1$, we have $\mathfrak{F}^G = \cup_n \mathfrak{F}^{K_n}$, so that the result follows from Proposition 1.1.2. □

The *Fell topology* on the family $\mathfrak{F}$ of closed sets (see Appendix C) is generated by open sets $\mathfrak{F}_G$ for $G \in \mathcal{G}$ and $\mathfrak{F}^K$ for $K \in \mathfrak{K}$. Therefore, the σ-algebra generated by $\mathfrak{F}_K$ for $K \in \mathfrak{K}$ coincides with the *Borel* σ-algebra $\mathfrak{B}(\mathfrak{F})$ generated by the Fell topology on $\mathfrak{F}$. It is possible to reformulate Definition 1.1.1 as follows.

Definition 1.1.1″ Assume that $\mathbb{E}$ is LCHS. A map $X: \Omega \mapsto \mathfrak{F}$ is called a random closed set if X is measurable with respect to the Borel σ-algebra on $\mathfrak{F}$ with respect to the Fell topology, i.e.

$$X^{-1}(\mathfrak{Y}) = \{\omega : X(\omega) \in \mathfrak{X}\} \in \mathfrak{A}$$

for each $\mathfrak{Y} \in \mathfrak{B}(\mathfrak{F})$.

Condition (1.1.1) can be reformulated as

$$X^{-1}(\mathfrak{F}_K) = \{\omega : X(\omega) \in \mathfrak{F}_K\} \in \mathfrak{A}. \tag{1.1.3}$$

It is easy to see that (1.1.3) implies the measurability of a number of further events, e.g., $\{X \cap G \neq \emptyset\}$ for every $G \in \mathcal{G}$ as confirmed by Proposition 1.1.2, $\{X \cap F \neq \emptyset\}$ and $\{X \subset F\}$ for every $F \in \mathfrak{F}$. Letting $F = \mathbb{E}$ yields that $\{X \cap \mathbb{E} \neq \emptyset\} = \{X \neq \emptyset\}$ is also measurable.

If two random closed sets X and Y share the same distribution, then we write $X \overset{\mathrm{d}}{\sim} Y$. This means $\mathbf{P}\{X \in \mathfrak{Y}\} = \mathbf{P}\{Y \in \mathfrak{Y}\}$ for every measurable family of closed sets $\mathfrak{Y} \in \mathfrak{B}(\mathfrak{F})$. In the following we see that this is the case if and only if $\mathbf{P}\{X \cap K \neq \emptyset\} = \mathbf{P}\{Y \cap K \neq \emptyset\}$ for all compact sets K (assuming $\mathbb{E}$ is LCHS).

Examples of Random Closed Sets

Example 1.1.4 (Singleton) If ξ is a random element in $\mathbb{E}$ (measurable with respect to the Borel σ-algebra on $\mathbb{E}$), then the singleton $X = \{\xi\}$ is a random closed set.

Example 1.1.5 (Half-line) If ξ is a random variable, then $X = (-\infty, \xi]$ is a random closed set on the line $\mathbb{E} = \mathbb{R}$. Indeed, $\{X \cap K \neq \emptyset\} = \{\xi \geq \inf K\}$ is a measurable event for every $K \subset \mathbb{E}$. Along the same lines, $X = (-\infty, \xi_1] \times \cdots \times (-\infty, \xi_d]$ is a random closed subset of $\mathbb{R}^d$ if $(\xi_1, \ldots, \xi_d)$ is a d-dimensional random vector.

Example 1.1.6 (Random interval) If ξ and η are two random variables in $\mathbb{R}$ such that $\xi \leq \eta$ a.s., then the random interval $X = [\xi, \eta]$ is a random closed set. This can be checked directly as $\{X \cap K = \emptyset\} = \{\eta < \inf K\} \cup \{\xi > \sup K\}$.

Example 1.1.7 (Random triangle and random ball) If ξ_1, ξ_2, ξ_3 are three random vectors in $\mathbb{R}^d$, then the triangle with vertices ξ_1, ξ_2 and ξ_3 is a random closed set. If ξ is a random vector in $\mathbb{R}^d$ and η is a non-negative random variable, then the random ball $B_\eta(\xi)$ of radius η centred at ξ is a random closed set (Fig. 1.1.1). While it is possible to deduce this directly from Definition 1.1.1, it is easier to refer to general results established later on in Theorem 1.3.25.

Example 1.1.8 (Random line) Let (ξ, η) be a random point from $\mathbb{R}_+ \times [0, 2\pi)$. The line in $\mathbb{R}^2$ orthogonal to the direction given by η and located at distance ξ from the origin is a random closed set. It is obtained by mapping a random singleton to a line using a set-valued map, see Appendix E. Many other random sets are defined in this way as $M(\xi)$, applying a set-valued function $M: \mathbb{R}^m \mapsto \mathfrak{F}$ to a random vector in $\mathbb{R}^m$.

Example 1.1.9 (Random set in finite space) Let $\mathbb{E} = \{x_1, \ldots, x_n\}$ be a finite space of cardinality n. Equipped with the discrete topology (so that all its subsets are closed and open at the same time) it is an LCHS space. Then X is a random set in $\mathbb{E}$ if and only if the vector $(\mathbf{1}_{x_1 \in X}, \ldots, \mathbf{1}_{x_n \in X})$ of indicators is a random vector with values in $\{0, 1\}^n$.

Example 1.1.10 (Levels and excursions of stochastic process) Let ζ_x, $x \in \mathbb{E}$, be a real-valued stochastic process on $\mathbb{E}$ with continuous sample paths. Then its level set $X = \{x \in \mathbb{E} : \ \zeta_x = t\}$ is a random closed set for every $t \in \mathbb{R}$. Indeed,

$$\{X \cap K = \emptyset\} = \left\{\inf_{x \in K} \zeta_x > t\right\} \cup \left\{\sup_{x \in K} \zeta_x < t\right\}$$

is measurable. Similarly, $\{x : \ \zeta_x \leq t\}$ and $\{x : \ \zeta_x \geq t\}$ are random closed sets. If the stochastic process ζ_x, $x \in \mathbb{E}$, is not necessarily continuous, then these random sets are graph measurable and not necessarily closed, see Example 1.3.33.

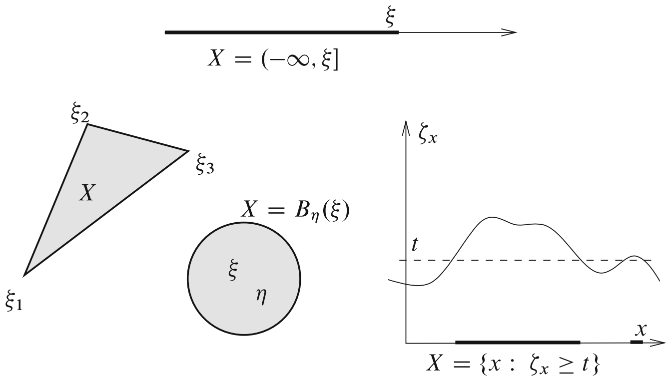

Fig. 1.1.1 Simple examples of random closed sets

Random Variables Associated with Random Closed Sets

Example 1.1.11 (Indicator) For every $x \in \mathbb{E}$, the indicator $\mathbf{1}_X(x)$ (equal to 1 if $x \in X$ and to zero otherwise) is a random variable.

Example 1.1.12 (Norm) The norm

$$\|X\| = \sup\{\|x\| : x \in X\}$$

of an almost surely non-empty random closed set X in $\mathbb{E} = \mathbb{R}^d$ is a random variable (with possibly infinite values). The event $\{\|X\| > t\}$ means that X hits the open set G, being the complement of the closed ball of radius t centred at the origin.

Example 1.1.13 Let ρ be a metric on $\mathbb{E}$. For each $x \in \mathbb{E}$, the *distance function*

$$\rho(x, X) = \inf\{\rho(x, y) : y \in X\}, \quad x \in \mathbb{E},$$

is a random variable with values in $[0, \infty]$, where the value ∞ arises if X is empty. Indeed, $\{\rho(x, X) \le t\} = \{B_t(x) \cap X \neq \emptyset\}$. Considered as a function of x, the distance function is a continuous stochastic process.

Example 1.1.14 (Measure of random set) If μ is a σ-finite Borel measure on $\mathbb{E}$, then $\mu(X)$ is a random variable. This follows directly from Fubini's theorem since $\mu(X) = \int \mathbf{1}_X(x)\mu(\mathrm{d}x)$, see Sect. 1.5.3. If $\mathbb{E}$ is a finite space and $\mu(A)$ is the number of points in A, then the random variable

$$\mu(X) = \mathrm{card}(X) = \sum_{x \in \mathbb{E}} \mathbf{1}_{x \in X}$$

is the cardinality of X. The same applies for the case of a countable $\mathbb{E}$, however, then the cardinality of X may become infinite.

Proposition 1.1.15 *Let $\mathbb{E}$ be LCHS.*

(i) *The number* $\mathrm{card}(X \cap B)$ *of points in $X \cap B$ is a random variable (with possibly infinite values) for each Borel set B in $\mathbb{E}$.*
(ii) *The number of connected components of a random closed set X is a random variable (with possibly infinite values).*

Proof. **(i)** It suffices to prove that $\mathrm{card}(X \cap G)$ is a random variable for all open G. Then $\mathrm{card}(X \cap G)$ is the supremum of $\sum_i \mathbf{1}_{X \cap G_i \neq \emptyset}$ over all $n \geq 1$ and disjoint $G_1, G_2, \dots, G_n$ from the countable base of the topology and such that $G_i \subset G$ for all i. It suffices to note that $\mathbf{1}_{X \cap G_i \neq \emptyset}$ is a random variable, and the supremum is taken over a countable family.
(ii) Let $\{K_n, n \geq 1\}$ be a sequence of compact sets that grows to $\mathbb{E}$. The number of connected components of X is the limit of the number of connected components of $X \cap K_n$, so that it suffices to prove the result for random sets in compact spaces. The number of connected components in X is at most n if X is covered by the union of n disjoint open sets. Since $\mathbb{E}$ has a countable base and X is compact, these open disjoint open sets $G_1, \dots, G_n$ may be chosen as finite unions of sets from the base. Finally, note that the event $X \subset G_1 \cup \dots \cup G_n$ is measurable. □

A closed set F has *reach* at least $r > 0$ if each point y from its r-envelope F^r (see (A.1)) admits the unique point $y \in F$ that is nearest to x. The set F is said to be of *positive reach* if the supremum of such r (called the reach of F) is strictly positive. Each convex set has infinite (hence, positive) reach.

Proposition 1.1.16 *The reach of a random closed set in $\mathbb{R}^d$ is a random variable.*

Proof. It suffices to show that the family of sets of reach at least r is closed in $\mathfrak{F}$ and so is measurable. Let $\{F_n, n \geq 1\}$ be a sequence of sets of reach at least r, and let $F_n \to F$ in the Fell topology. By Theorems C.7 and C.14, the distance functions $\rho(x, F_n)$ converge to $\rho(x, F)$ uniformly for all x in any compact set. By Federer [264, Th. 4.13], the limiting set F has reach at least r. □

1.1.2 Capacity Functionals

Definition

The distribution of a random closed set X is determined by $\mathbf{P}(\mathfrak{Y}) = \mathbf{P}\{X \in \mathfrak{Y}\}$ for all $\mathfrak{Y} \in \mathfrak{B}(\mathfrak{F})$. The particular choice of $\mathfrak{Y} = \mathfrak{F}_K$ and $\mathbf{P}\{X \in \mathfrak{F}_K\} = \mathbf{P}\{X \cap K \neq \emptyset\}$ is useful since the families $\mathfrak{F}_K$, $K \in \mathfrak{K}$, generate the Borel σ-algebra $\mathfrak{B}(\mathfrak{F})$.

Definition 1.1.17 (Capacity functional) The functional $T_X \colon \mathcal{K} \mapsto [0,1]$ given by

$$T_X(K) = \mathbf{P}\{X \cap K \neq \emptyset\}, \quad K \in \mathcal{K}, \tag{1.1.4}$$

is said to be the *capacity functional* of X. We write $T(K)$ instead of $T_X(K)$ where no ambiguity occurs.

Example 1.1.18 (Capacity functionals of simple random sets)
(i) If $X = \{\xi\}$ is a random singleton, then $T_X(K) = \mathbf{P}\{\xi \in K\}$, so that the capacity functional is the probability distribution of ξ. If X is $\{\xi\}$ with probability p and otherwise is equal to the whole space $\mathbb{E}$, then $T_X(K) = p\mathbf{P}\{\xi \in K\} + (1-p)\mathbf{1}_{K \neq \emptyset}$.
(ii) Let $X = \{\xi_1, \xi_2\}$ be the set formed by two independent identically distributed random elements in $\mathbb{E}$ (X is a singleton if $\xi_1 = \xi_2$). Then $T_X(K) = 1 - (1 - \mathbf{P}\{\xi_1 \in K\})^2$. For instance, if ξ_1 and ξ_2 are the numbers shown by two dice, then $X \subset \{1, 2, \dots, 6\}$ and $T_X(\{6\})$ is the probability that at least one die shows six.
(iii) Let $X = (-\infty, \xi]$ be a random closed set in $\mathbb{R}$, where ξ is a random variable. Then $T_X(K) = \mathbf{P}\{\xi > \inf K\}$ for all $K \in \mathcal{K}$.
(iv) If $X = \{x \in \mathbb{E} : \ \zeta_x \geq t\}$ for $t \in \mathbb{R}$ and a real-valued sample continuous stochastic process ζ_x, $x \in \mathbb{E}$, then $T_X(K) = \mathbf{P}\{\sup_{x \in K} \zeta_x \geq t\}$. The capacity functional at $K = \{x\}$ is $\mathbf{P}\{\zeta_x \geq t\}$.
(v) If $X = \{t \geq 0 : \ w_t = 0\}$ is the set of zeros for the standard Brownian motion w_t, then

$$T_X([a, b]) = \frac{2}{\pi} \arccos \sqrt{a/b}$$

by the arcsine law, see, e.g., Kallenberg [443, Th. 13.16].

It follows immediately from the definition of $T = T_X$ that

$$T(\emptyset) = 0, \tag{1.1.5}$$

and

$$0 \leq T(K) \leq 1, \quad K \in \mathcal{K}. \tag{1.1.6}$$

It should be noted that $\mathbf{P}\{X \cap \mathbb{E} \neq \emptyset\} = \mathbf{P}\{X \neq \emptyset\}$ (which can be viewed as the value $T(\mathbb{E})$ of the capacity functional extended to possibly non-compact sets) may be strictly less than one.

Since $\mathfrak{F}_{K_n} \downarrow \mathfrak{F}_K$ as $K_n \downarrow K$, the continuity property of the probability measure $\mathbf{P}$ implies that T is *upper semicontinuous* (see Proposition E.12), i.e.

$$T(K_n) \downarrow T(K) \quad \text{as} \;\; K_n \downarrow K \text{ in } \mathcal{K}. \tag{1.1.7}$$

Properties (1.1.5) and (1.1.7) mean that T is a (topological) precapacity that can be extended to the family of all subsets of $\mathbb{E}$ as described in Appendix G.

Complete Alternation

It is easy to see that the capacity functional T is *monotone*, i.e.

$$T(K_1) \le T(K_2) \quad \text{if } K_1 \subset K_2.$$

Moreover, T satisfies a stronger monotonicity property described below. To each functional T defined on a family of (compact) sets we can associate the following *successive differences*:

$$\Delta_{K_1} T(K) = T(K) - T(K \cup K_1), \tag{1.1.8}$$

$$\begin{aligned} \Delta_{K_n} \cdots \Delta_{K_1} T(K) &= \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K) \\ &\quad - \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K \cup K_n), \quad n \ge 2. \end{aligned} \tag{1.1.9}$$

Note that $\Delta_{K_n} \cdots \Delta_{K_1} T(K)$ is invariant under permutations of $K_1, \dots, K_n$. If T is the capacity functional of X, then

$$\begin{aligned} \Delta_{K_1} T(K) &= \mathbf{P}\{X \cap K \ne \emptyset\} - \mathbf{P}\{X \cap (K \cup K_1) \ne \emptyset\} \\ &= -\mathbf{P}\{X \cap K_1 \ne \emptyset,\ X \cap K = \emptyset\}. \end{aligned}$$

Applying this argument consecutively yields an important relationship between the higher-order successive differences and the distribution of X

$$\begin{aligned} -\Delta_{K_n} \cdots \Delta_{K_1} T(K) &= \mathbf{P}\{X \cap K = \emptyset,\ X \cap K_i \ne \emptyset,\ i = 1, \dots, n\} \\ &= \mathbf{P}\left\{X \in \mathfrak{F}^K_{K_1,\dots,K_n}\right\}, \end{aligned} \tag{1.1.10}$$

where

$$\begin{aligned} \mathfrak{F}^K_{K_1,\dots,K_n} &= \{F \in \mathfrak{F} :\ F \cap K = \emptyset,\ F \cap K_1 \ne \emptyset, \dots, F \cap K_n \ne \emptyset\} \\ &= \mathfrak{F}^K \cap \mathfrak{F}_{K_1} \cap \cdots \cap \mathfrak{F}_{K_n}, \end{aligned}$$

see Fig. 1.1.2. In particular, (1.1.10) implies

$$\Delta_{K_n} \cdots \Delta_{K_1} T(K) \le 0 \tag{1.1.11}$$

for all $n \ge 1$ and $K, K_1, \dots, K_n \in \mathfrak{K}$. Equation (1.1.11) establishes the *complete alternation* property of the capacity functional T, see Definition 1.1.23.

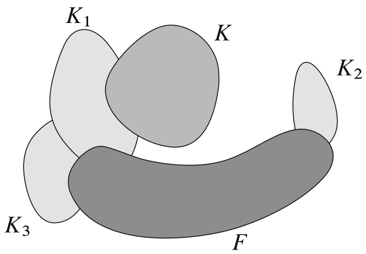

Fig. 1.1.2 A set F from $\mathfrak{F}^K_{K_1,K_2,K_3}$: it misses K and hits each of K_1, K_2, K_3

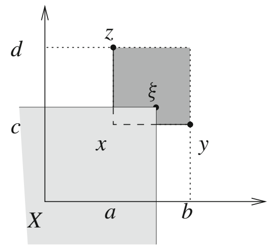

Fig. 1.1.3 Random closed set from Example 1.1.19(ii)

Example 1.1.19 (Higher-order differences)
(i) Let $X = \{\xi\}$ be a random singleton with distribution $\mathbf{P}$. Then

$$-\Delta_{K_n} \cdots \Delta_{K_1} T(K) = \mathbf{P}\left\{\xi \in (K_1 \cap \cdots \cap K_n \cap K^{\mathsf{c}})\right\}.$$

(ii) Let $X = (-\infty, \xi_1] \times (-\infty, \xi_2]$ be a random closed set in the plane $\mathbb{R}^2$. Then $-\Delta_{\{x\}} T(\{y, z\})$ for $x = (a, c)$, $y = (b, c)$, $z = (a, d)$ is the probability that ξ lies in the rectangle $[a, b) \times [c, d)$, see Fig. 1.1.3.
(iii) Let $X = \{x : \ \zeta_x \geq 0\}$ for a continuous real-valued random function ζ on $\mathbb{E}$. Then

$$-\Delta_{K_n} \cdots \Delta_{K_1} T(K) = \mathbf{P}\left\{\sup_{x \in K} \zeta_x < 0, \ \sup_{x \in K_i} \zeta_x \geq 0, \ i = 1, \ldots, n\right\}.$$

The properties of the capacity functional T resemble those of the cumulative distribution function of random vectors. The upper semicontinuity property (1.1.7) is similar to right-continuity, and (1.1.11) generalises the monotonicity concept. The complete alternation property (1.1.11) corresponds to the non-negativity of probability contents of parallelepipeds. While for d-dimensional random vectors it suffices to check the successive differences up to order d, all orders are needed for random sets.

In contrast to measures, the functional T is not additive, but only *subadditive*, i.e.

$$T(K_1 \cup K_2) \leq T(K_1) + T(K_2) \tag{1.1.12}$$

for all compact sets K_1 and K_2.

Example 1.1.20 (Non-additive capacity functional) If $X = B_r(\xi)$ is the ball of radius r centred at a random point ξ in $\mathbb{R}^d$, then $T_X(K) = \mathbf{P}\{\xi \in K^r\}$ is not additive, since the r-envelopes K_1^r and K_2^r are not necessarily disjoint for disjoint K_1 and K_2.

Inequality (1.1.11) for $n = 2$ turns into

$$T(K) + T(K \cup K_1 \cup K_2) \leq T(K \cup K_1) + T(K \cup K_2), \quad K, K_1, K_2 \in \mathcal{K}, \tag{1.1.13}$$

which yields (1.1.12) if $K = \emptyset$. By letting K_1 be $K \cup K_1$ and K_2 be $K \cup K_2$ and using the monotonicity of T, (1.1.13) is equivalent to

$$T(K_1 \cap K_2) + T(K_1 \cup K_2) \leq T(K_1) + T(K_2), \quad K_1, K_2 \in \mathcal{K}, \tag{1.1.14}$$

meaning that T is *concave*, also called strongly subadditive.

Extension of the Capacity Functional

As explained in Appendix G, a capacity φ defined on compact sets in an LCHS space can be naturally extended to the family $\mathcal{P} = \mathcal{P}(\mathbb{E})$ of all subsets so that it preserves alternation or the monotonicity properties enjoyed by φ. In its application to capacity functionals of random closed sets, put

$$T^*(G) = \sup\{T(K) : K \in \mathcal{K}, K \subset G\}, \quad G \in \mathcal{G}, \tag{1.1.15}$$

and

$$T^*(M) = \inf\{T^*(G) : G \in \mathcal{G}, G \supset M\}, \quad M \in \mathcal{P}. \tag{1.1.16}$$

Theorem 1.1.21 (Consistency of extension)

(i) *$T^*(K) = T(K)$ for each $K \in \mathcal{K}$.*

(ii) *For all Borel sets B in $\mathbb{E}$,*

$$T^*(B) = \sup\{T(K) : K \in \mathcal{K}, K \subset B\}.$$

(iii) *The functional T^* is completely alternating on the family of all subsets of $\mathbb{E}$.*

Proof. The first statement follows from the upper semicontinuity of T. Note that $T^*(K)$ is a limit of $T^*(G_n)$ for a sequence of open sets $G_n \downarrow K$. By choosing $K_n \in \mathcal{K}$ such that $K \subset K_n \subset G_n$ we deduce that $T(K_n) \downarrow T^*(K)$, while at the same time $T(K_n) \downarrow T(K)$ since T is upper semicontinuous. The second statement is a corollary from the more intricate Choquet capacitability theorem, see Theorem G.2. The complete alternation of the extension follows from Proposition G.6. $\square$

Since the extension T^* coincides with T on $\mathcal{K}$, in the following we use the same notation T to denote the extension, i.e. $T(G)$ or $T(B)$ denotes the values of the extended T on open G and Borel B. While T is upper semicontinuous on compact sets, its extension satisfies $T(G_n) \uparrow T(G)$ for any sequence of open sets $G_n \uparrow G$.

Theorem 1.1.22 *The extended capacity functional satisfies*

$$T(B) = \mathbf{P}\{X \cap B \neq \emptyset\}$$

for all Borel B.

Proof. By letting $K_n \uparrow G \in \mathcal{G}$, the continuity property of probability measures yields that $T(G) = \mathbf{P}\{X \cap G \neq \emptyset\}$ for all $G \in \mathcal{G}$. The event $\{X \cap B \neq \emptyset\}$ is measurable by Theorem 1.3.3. By (1.1.16), it is possible to find a decreasing sequence of open sets $G_n \supset B$ such that $T(G_n) \downarrow T(B)$. By Theorem 1.1.21(ii), there is an increasing sequence of compact sets $K_n \subset B$ such that $T(K_n) \uparrow T(B)$. It suffices to note that

$$T(K_n) \leq \mathbf{P}\{X \cap B \neq \emptyset\} \leq T(G_n). \qquad \square$$

The countable subadditivity property of probability measures yields that the capacity functional is *countably subadditive* on Borel sets, that is,

$$T\left(\cup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} T(B_i) \tag{1.1.17}$$

for all $B_1, B_2, \ldots \in \mathfrak{B}(\mathbb{E})$.

Complete Alternation and Monotonicity of General Functionals

Because of the importance of the upper semicontinuity property (1.1.7) and the complete alternation (1.1.11), it is natural to consider general functionals that satisfy these properties without immediate reference to distributions of random closed sets.

Definition 1.1.23 (Completely alternating and completely $\cup$-monotone functionals) Let $\mathfrak{D}$ be a family of sets which is closed under finite unions (so that $M_1 \cup M_2 \in \mathfrak{D}$ if $M_1, M_2 \in \mathfrak{D}$). A real-valued functional φ defined on $\mathfrak{D}$ is said to be

(i) *completely alternating* or completely $\cup$-alternating (notation $\varphi \in \mathsf{A}(\mathfrak{D})$ or $\varphi \in \mathsf{A}_\cup(\mathfrak{D})$) if

$$\Delta_{K_n} \cdots \Delta_{K_1} \varphi(K) \leq 0, \quad n \geq 1,\ K, K_1, \ldots, K_n \in \mathfrak{D}\,; \tag{1.1.18}$$

(ii) *completely* $\cup$*-monotone* (notation $\varphi \in \mathsf{M}_\cup(\mathfrak{D})$) if

$$\Delta_{K_n} \cdots \Delta_{K_1} \varphi(K) \geq 0, \quad n \geq 1, \ K, K_1, \dots, K_n \in \mathfrak{D}.$$

Definition 1.1.23(i) with $\mathfrak{D} = \mathcal{K}$ and $\varphi = T_X$ corresponds to the complete alternation property of the capacity functional. Definition 1.1.23 complies with Definition I.5 applied to the *semigroup* $\mathfrak{D}$ with the union being the semigroup operation, Therefore, it is possible to use the results of Appendix I within this context. Theorem I.8 states that $\varphi \in \mathsf{A}_\cup(\mathcal{K})$ if and only if $\mathrm{e}^{-t\varphi} \in \mathsf{M}_\cup(\mathcal{K})$ for all $t > 0$. Let us formulate one particularly important corollary of this fact.

Proposition 1.1.24 *If φ is a completely alternating non-negative functional, then* $1 - \mathrm{e}^{-\varphi(K)}$ *is a completely alternating functional with values in* $[0, 1)$.

Proposition 1.1.24 is often used to construct a capacity functional from a completely alternating upper semicontinuous functional that may take values greater than one.

Example 1.1.25 Every measure μ is a completely alternating functional, since

$$-\Delta_{K_n} \cdots \Delta_{K_1} \mu(K) = \mu((K_1 \cup \cdots \cup K_n) \setminus K) \geq 0.$$

In particular, $\Delta_{K_1} \mu(K) = -\mu(K_1)$ if K and K_1 are disjoint.

If (1.1.18) holds for all $n \leq k$ and some natural number k, then the functional φ is called k-alternating. In particular, φ is increasing if and only if it is 1-alternating, that is,

$$\Delta_{K_1} \varphi(K) = \varphi(K) - \varphi(K \cup K_1) \leq 0.$$

Since

$$\Delta_{K_2} \Delta_{K_1} \varphi(K) = \varphi(K) - \varphi(K \cup K_1) - \varphi(K \cup K_2) + \varphi(K \cup K_1 \cup K_2),$$

(1.1.18) for $n = 2$ turns into

$$\varphi(K) + \varphi(K \cup K_1 \cup K_2) \leq \varphi(K \cup K_1) + \varphi(K \cup K_2). \tag{1.1.19}$$

In particular, if $\mathfrak{D}$ is closed under finite intersections, contains the empty set, and $\varphi(\emptyset) = 0$, then letting $K = \emptyset$ in (1.1.19) yields that

$$\varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2), \tag{1.1.20}$$

meaning that φ is subadditive. For an increasing φ, inequality (1.1.19) is equivalent to

$$\varphi(K_1 \cap K_2) + \varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2) \tag{1.1.21}$$

for all K_1 and K_2. A functional φ satisfying (1.1.21) is called *concave* or strongly subadditive. Functionals satisfying the reverse inequality in (1.1.21) are called *convex* or strongly superadditive. Furthermore, φ is called *2-alternating* if $\Delta_{K_1}\varphi(K)$ and $\Delta_{K_2}\Delta_{K_1}\varphi(K)$ are non-positive for all $K, K_1, K_2 \in \mathfrak{D}$. Therefore, φ is 2-alternating if it is both concave and monotone.

Another natural semigroup operation on sets is intersection, which leads to other concepts of alternating and monotone functionals. Similarly to the definition of $\Delta_{K_n} \cdots \Delta_{K_1}\varphi(K)$, we introduce the following successive differences

$$\nabla_{K_1}\varphi(K) = \varphi(K) - \varphi(K \cap K_1), \tag{1.1.22}$$

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) = \nabla_{K_{n-1}} \cdots \nabla_{K_1}\varphi(K) - \nabla_{K_{n-1}} \cdots \nabla_{K_1}\varphi(K \cap K_n), \quad n \geq 2. \tag{1.1.23}$$

The following definition is a direct counterpart of Definition 1.1.23.

Definition 1.1.26 (Completely $\cap$-alternating and completely monotone functionals) Let $\mathfrak{D}$ be a family of sets which is closed under finite intersections. A real-valued functional φ defined on $\mathfrak{D}$ is said to be

(i) *completely $\cap$-alternating* (notation $\varphi \in \mathsf{A}_\cap(\mathfrak{D})$) if

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) \leq 0, \quad n \geq 1,\ K, K_1, \ldots, K_n \in \mathfrak{D}\,;$$

(ii) *completely monotone* or completely $\cap$-monotone (notation $\varphi \in \mathsf{M}(\mathfrak{D})$ or $\varphi \in \mathsf{M}_\cap(\mathfrak{D})$) if

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) \geq 0, \quad n \geq 1,\ K, K_1, \ldots, K_n \in \mathfrak{D}.$$

When saying that φ is completely alternating we always mean that φ is completely $\cup$-alternating, while calling φ completely monotone means that φ is completely $\cap$-monotone.

If $\mathfrak{D}$ is closed both under finite unions and under finite intersections, the complete alternation condition can be equivalently formulated as

$$\varphi(\bigcap_{i=1}^{n} K_i) \leq \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{\mathrm{card}(J)+1}\varphi(\bigcup_{i \in J} K_i) \tag{1.1.24}$$

for all $n \geq 1$ and $K_1, \ldots, K_n \in \mathfrak{D}$. The complete monotonicity condition is equivalent to

$$\varphi(\bigcup_{i=1}^{n} K_i) \geq \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{\mathrm{card}(J)+1}\varphi(\bigcap_{i \in J} K_i). \tag{1.1.25}$$

Proposition 1.1.27 *Let $\varphi\colon \mathfrak{D} \mapsto [0,1]$. Then,*

(i) $\varphi \in \mathsf{A}_\cup(\mathfrak{D})$ *if and only if, for any fixed $L \in \mathfrak{D}$,*

$$-\Delta_L\varphi(K) = \varphi(K \cup L) - \varphi(K) \in \mathsf{M}_\cup(\mathfrak{D})\,;$$

(ii) $\varphi \in \mathsf{A}_\cap(\mathfrak{D})$ *if and only if, for any fixed $L \in \mathfrak{D}$,*

$$-\nabla_L\varphi(K) = \varphi(K \cap L) - \varphi(K) \in \mathsf{M}_\cap(\mathfrak{D}).$$

(iii) *Let $\varphi\colon \mathfrak{D} \mapsto [0,1]$. Then $\varphi \in \mathsf{A}_\cup(\mathfrak{D})$ (respectively, $\varphi \in \mathsf{A}_\cap(\mathfrak{D})$) if and only if $\tilde{\varphi}(K) \in \mathsf{M}_\cap(\mathfrak{D}')$ (respectively, $\tilde{\varphi}(K) \in \mathsf{M}_\cup(\mathfrak{D}')$) for the* dual *functional*

$$\tilde{\varphi}(K) = 1 - \varphi(K^{\mathsf{c}}), \quad K^{\mathsf{c}} \in \mathfrak{D}, \tag{1.1.26}$$

defined on the family $\mathfrak{D}' = \{K^{\mathsf{c}} :\ K \in \mathfrak{D}\}$ of complements of the sets from $\mathfrak{D}$.

Proof. (i) It suffices to note that

$$\Delta_{K_n} \dots \Delta_{K_1}(-\Delta_L\varphi(K)) = -\Delta_L\Delta_{K_n} \dots \Delta_{K_1}\varphi(K)$$

with a similar relationship valid for the successive differences based on intersections. Statement (ii) is proved similarly. The proof of (iii) is a matter of verification that

$$\Delta_{K_n} \cdots \Delta_{K_1}\tilde{\varphi}(K) = -\nabla_{K_n^{\mathsf{c}}} \cdots \nabla_{K_1^{\mathsf{c}}}\varphi(K^{\mathsf{c}}).$$ □

Complete Alternation and Positive Definiteness

The family $\mathcal{K}$ becomes an abelian semigroup if equipped with the union operation. This semigroup is idempotent and also 2-divisible, meaning that each element can be represented as the sum of two identical elements, that is, $K = K \cup K$. In this case, the family of completely alternating functionals coincides with the family of *negative definite* functionals, and the family of completely monotone functionals is the same as the family of *positive definite* functionals, see Berg, Christensen and Ressel [92, Cor. 4.6.8] and Theorem I.6. This fact is presented in the following theorem.

Theorem 1.1.28 *A functional $\varphi\colon \mathcal{K} \mapsto \mathbb{R}_+$ is completely alternating if and only if*

$$\sum_{i=1}^{n} c_i c_j \varphi(K_i \cup K_j) \le 0$$

for all $n \ge 2$, $K_1, \dots, K_n \in \mathcal{K}$, and $c_1, \dots, c_n \in \mathbb{R}$ such that $\sum c_i = 0$.