3D Modeling of Nonlinear Wave Phenomena on Shallow Water Surfaces

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Preface

How mesmerizing is the beauty of the waves approaching the seashore against a background of the sunset: they try to catch up with each other in a continuous cycle of water flow, then they subside, then intensify, rolling up on the shore, crashing into a sparkling foam, creating an endless symphony of surf. You can endlessly admire this landscape, which has existed for billions of years, from the time when there were no living beings on the planet Earth. Also, primeval ocean waves wash ashore, as is happening now in the presence of a person watching this picture. These waves have attracted the attention of artists and researchers for more than a century. Despite their beauty and simplicity, however, they are not always easy to describe. Moreover, to verify the plausibility of the created model, special knowledge is not necessarily required. It’s enough to go to the beach, and everything will become clear.

At the same time, neglecting the power of this beauty can lead to devastating consequences in storm surges and earthquakes. Therefore, the study of waves on the sea surface is not an easy task, and attempts are made in this work to describe and simulate some wave events on the surface of the aquatic environment. By their nature, these waves are inherently nonlinear, although some approximations may be considered linear. Consequently, the most appropriate theory of surface wave description is nonlinear theory.

This book presents the work done by the author for the research and modeling of nonlinear wave activities on the shallow water surface. An attempt was made to describe the run-up of surface waves to various coastal formations in shallow waters. Photographic illustrations of wave activities on the shallow water surface, made by the author, are also provided to illustrate the work.

I want to express my appreciation to my teachers, and promote a love for mathematics, art, and beauty.

Iftikhar B. Abbasov
Introduction

In the context of the study of the ecosystems of the shallow coastal areas of the world’s oceans, physical phenomena occurring on the surface of the aquatic environment play an important role. These phenomena, like all natural phenomena, are complex and nonlinear. Therefore, this leads to the nonlinear mathematical models of the actual processes.

The theory of wave motion fluids is a classical section of hydrodynamics and has a three-hundred-year history. The interest in wave activities on the surface of the fluid could be explained by the prevalence and accessibility of this physical phenomenon. Despite a great deal of research, the theory of wave fluid movements is still incomplete.

Of great importance is the matter of researching and modeling the wave activities at shallow water and the impact of surface gravity waves to coast formations and hydrotechnical structures. Therefore, the question of 3D modeling of the distribution, run-up and refraction of nonlinear surface waves can play an important role in monitoring and forecasting the sustainable development of the ecosystems of these areas.

The results of the research and numerical modeling of the dynamic of nonlinear surface gravity waves at shallow water are introduced in this work. Corresponding equations of mathematical physics and methods of mathematical modeling are used for describing and modeling.

Analytical descriptions of these nonlinear wave activities often use different modifications of the shallow water equations. For the numerical modeling, shallow water equations are also used in a 1D case. 2D and 3D numerical modeling of nonlinear surface gravity waves to beach approaches are based on Navier-Stokes equations. Navier-Stokes equations allow for both nonlinear effects and turbulent processes to be considered in the incompressible fluid.

Therefore, appropriate nonlinear waves of hydrodynamic equations will be used to adequately model nonlinear wave activities in shallow water conditions.
1

Equations of Hydrodynamics

1.1 Features of the Problems in the Formulation of Mathematical Physics

When examining a physical process, the scientist needs to describe it in mathematical terms. A mathematical description or a process modeling could be quite varied. Mathematical modeling does not investigate the actual physical process itself, and some of its models are the ideal process written in the form of mathematics. The mathematical model should preserve the basic features of the actual physical process and, at the same time, should be simple enough to be solved by known methods. In the future, the consistency of the mathematical model with the actual process needs to be tested.

Many ways of mathematically describing physical processes lead to differential equations with private derivatives, and in some cases to Integro-differential equations. It is this group of tasks that is assigned the term mathematical physics, and the methods of solving them are referred to as mathematical physics methods.
The subject of mathematical physics is the mathematical theory of physical phenomena. The wide distribution of mathematical physics is connected to the commonality of mathematical models based on fundamental laws of nature: the laws of mass, energy, charge conservation, kinetic momentum. This results in the same mathematical models describing the physical phenomena of different natures.

Mathematical physics usually examines processes in a certain spatial area filled with a continuous material environment called the solid environment. Values that describe the state of the environment and the physical processes that occur in it depend on the spatial coordinates and time. In general, mathematical physics models describe the behavior of the system at three levels: the interaction of the system as a whole with the external environment; the interaction between the system's basic volumes and the properties of a single, basic system volume.

The interaction of the system with the external environment is the wording of the boundary conditions, i.e., the conditions at the border of the task area, which include in general the boundary and initial conditions. The second level describes the interaction of elementary volumes based on laws for the preservation of physical substances and their transfer in space. The third level corresponds to the establishment of the state equations of the environment, i.e., the creation of a mathematical model of the basic environment behavior.

The equations of mathematical physics emerged from the consideration of such essential physical tasks as the distribution of sound in gases, waves in liquids, heat in physical bodies. The phenomena of nuclear reaction, gravity, electromagnetic effects, the origin and evolution of the universe are being actively explored now. Mathematical models of these different physical phenomena lead to equations with private derivatives.

An equation with a private derivative is an equation that includes an unknown function that depends on several variables and its private derivatives. Dependence on many variables in an unknown function makes it much harder to solve equations with private derivatives. Very few of these equations are explicitly solved.

As a result of the development of computer technology, the role of computational methods in the approximation of mathematical physics has grown. However, the approximate analytical methods that make it possible to obtain the connection between the functions sought and the specified parameters of the task in question have not lost their importance.

A precise analytical solution to mathematical physics usually requires the integration of differential equations with private derivatives. These equations need to be integrated into a certain spatial-temporal area where
the desired functions are subjected to the specified boundary conditions. Therefore, a precise analytical solution to such equations is possible only in rare cases, which underscores the importance of approximation methods. Before we go into the methods of solving equations, consider classifying differential equations with private derivatives.

### 1.2 Classification of Linear Differential Equations with Partial Derivatives of the Second Order

Many problems of mathematical physics lead to linear differential equations of the second order. For an unknown function $u$, a linear differential equation of the second order, depending on two variables $x$ and $y$, has the following form [Aramanovich, 1969]:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = f(x, y) \quad (1.2.1)$$

We assume that all the coefficients of the equation are constant. Most differential equations of mathematical physics represent particular cases of the common equation (1.2.1).

L. Euler proved that any differential equation of the form (1.2.1) by replacing the independent variables $x$ and $y$ can be reduced to one of the following three types:

1. If, $AC - \frac{B^2}{4} > 0$, then, after introducing new independent variables $\xi$ and $\eta$ equation (1.2.1) takes the form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + D_1 \frac{\partial u}{\partial \xi} + E_1 \frac{\partial u}{\partial \eta} + F_1 u = f_1(\xi, \eta) \quad (1.2.2)$$

In this case, the equation is called *elliptic*. The simplest elliptic equation is the Laplace equation.

2. If, $AC - \frac{B^2}{4} < 0$, then equation (1.2.1) can be given the form

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + D_2 \frac{\partial u}{\partial \xi} + E_2 \frac{\partial u}{\partial \eta} + F_2 u = f_2(\xi, \eta) \quad (1.2.3)$$
Such an equation is called hyperbolic; the simplest example of this is the one-dimensional equation of free oscillations.

3. If, \( AC - \frac{B^2}{4} = 0 \), then equation (1.2.1) is reduced to the next:

\[
\frac{\partial^2 u}{\partial \xi^2} + D \frac{\partial u}{\partial \xi} + E \frac{\partial u}{\partial \eta} + F_3 u = f_3(\xi, \eta) \tag{1.2.4}
\]

This equation is called parabolic. An example of it is the equation of linear thermal conductivity.

The names of the equations are explained by the fact that in the study of the common equation of curves of the second order

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,
\]

it turns out that the curve represents:

- in the case \( AC - \frac{B^2}{4} > 0 \) – of an ellipse;
- in the case \( AC - \frac{B^2}{4} < 0 \) – of an hyperbole;
- in the case \( AC - \frac{B^2}{4} = 0 \) – of an parabola.

Finally, any equation of the form (1.2.1) can be reduced to one of the following canonical types:

\[
\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + cu = f \text{ (elliptical type)},
\]

\[
\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} + cu = f \text{ (hyperbolic type)},
\]

\[
\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial u}{\partial \eta} = f \text{ (parabolic type)},
\]

\( (c \text{ – constant number, } f \text{ – function of variables } x \text{ and } y) \).

Equations of hyperbolic and parabolic types arise most often when studying processes occurring in time (equations of oscillations, wave propagation, heat propagation, diffusion). In the one-dimensional case, one coordinate always participates \( x \) and time \( t \). Additional conditions for such tasks, divided into initial and boundary.
The initial conditions consist in setting for $t=0$ the values of the desired function $u$ and its derivative (in the hyperbolic case) or only the values of the function itself (in the parabolic case).

The boundary conditions for these problems lie in the fact that the values of the unknown function $u(x,t)$ are indicated at the ends of the coordinate change interval.

If the process proceeds in an infinite interval of variation of the coordinate $x$, then the boundary conditions disappear, and the problem is obtained only with initial conditions, or, as it is often called, the Cauchy problem.

If a problem is posed for a finite interval, then the initial and boundary conditions must be given. Then we speak of a mixed problem.

Equations of elliptic type arise usually in the study of stationary processes. The time $t$ does not enter into these equations, and both independent variables are the coordinates of the point. Such are the equations of the stationary temperature field, the electrostatic field, and the equations of many other physical problems. For problems of this type, only boundary conditions are set, that is, specifies the behavior of the unknown function on the contour area. This can be the Dirichlet problem, when the values of the function itself are given; the Neumann problem when the values of the normal derivative of the unknown function are given; and the problem, when a linear combination of the function is given on the contour, and its normal derivative.

In the basic problems of mathematical physics, it is physical considerations that prompt what additional conditions should be put in one or another problem in order to obtain a unique solution of it that corresponds to the nature of the process being studied.

In addition, it should be borne in mind that all the equations derived are of an idealized nature, that is, they reflect only the most essential features of the process. The functions entering into the initial and boundary conditions in physical problems are determined from experimental data and can be considered only approximately.

1.3 Nonlinear Equations of Fluid Dynamics

Linear integro-differential equations describe wave processes possessing the superposition property. In linear waves, the space-time spectral components of the wave fields propagate without distortion and do not interact with each other.

The linear medium is some idealized model for describing the real environment, and this is not always adequate. The applicability of the linear
medium model depends first of all on the magnitude of the ratio of the wave amplitude to the characteristic quantity that determines the properties of the medium. In a linear environment, the ratio of the wave amplitude to the characteristic value of the medium is assumed to be infinitesimal, as a result of which the wave equation becomes linear.

For a finite value of this ratio, it is necessary to take into account nonlinear terms in the wave equation. The inclusion of nonlinear terms in the wave equation leads to qualitatively new phenomena. If a monochromatic wave is fed to the input of such a system, then the nonlinearity leads to successive excitation of the time harmonics of the initial wave. The spreading of the frequency spectrum further distorts the shape of the initial sinusoidal wave profile.

In wave systems, the degree of nonlinear interaction is determined both by the considered local nonlinearity and by the ratio of the extent of the interaction region to the wavelength. The extent of the region of effective harmonic interaction largely depends on the dispersion and dissipation of the medium. The energy exchange between the harmonics depends on the phase relationship. In a medium without frequency dispersion, all waves run with the same velocities, and the phase relations remain in the process of propagation between the harmonics. This condition is called the phase matching condition. If the attenuation of the waves is small, even minor nonlinear effects can accumulate in proportion to the distance, and the wave will become unstable and breaking over time [Vinogradova, Rudenko, 1979].

In the case of a medium with dispersion, the phase velocities of the waves at different frequencies are different, so that the relations between the phases of the harmonics vary rapidly in space. In case of violation of phase matching nonlinear effects do not accumulate and energy transfer is negligible. Therefore, in the dispersive media there is no noticeable distortion of the shape of the wave profile.

Consider the nonlinear equations, which are often used in fluid dynamics, although they are found in many other areas of modern physics. Taking into account the analogy of nonlinear effects of any nature, one can create a model equation for a one-dimensional wave [Brekhovskikh, 1982], [Gabov, 1988]

\[
\frac{\partial u}{\partial t} + Lu = -\varepsilon u \frac{\partial u}{\partial x}. \quad (1.3.1)
\]

Here \( \varepsilon \ll 1 \) is the nonlinearity parameter; \( L \) – linear operator, corresponding to a certain dispersion of linear waves.
If \( L = c \partial / \partial x \) we obtain equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = -\varepsilon u \frac{\partial u}{\partial x} .
\] (1.3.2)

This is a nonlinear equation of acoustic type without dispersion and dissipation; its solution is Riemann invariants, leading to different propagation velocities of the compression and extension regions.

If \( L = c \frac{\partial}{\partial x} + \beta \frac{\partial^3}{\partial x^3} \) we obtain equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = -\varepsilon u \frac{\partial u}{\partial x} .
\] (1.3.3)

This equation of Korteweg and de Vries for nonlinear media with dispersion describes surface waves in shallow water and is a particular case of a more common shallow water equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial x} + \frac{1}{3} H \frac{\partial^3 u}{\partial x^3} = 0 .
\] (1.3.4)

Equation (1.3.4) is called the Boussinesq equation; it describes nonlinear waves of small amplitude, moving both to the left and to the right. For the case of two-dimensional waves in shallow water, the Kadomtsev-Petviashvili equation is used

\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + \frac{3c}{2H} u \frac{\partial u}{\partial x} + \frac{1}{6} cH^2 \frac{\partial^3 u}{\partial x^3} \right) + \frac{1}{2} c \frac{\partial^2 u}{\partial y^2} = 0 .
\] (1.3.5)

Surface wave profiles in deep water are described by the cubic Schrödinger equation

\[
i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \nu |u|^2 u = 0 .
\] (1.3.6)

Schrödinger equation relates to the parabolic type with a space variable. A description of the surface waves in shallow water with arbitrary dispersion used the Whitham equation with integral operator:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \int_{-\infty}^{+\infty} K(x-s) u(s,t) ds = 0 .
\] (1.3.7)
To study the effects of viscosity in fluids model equation is the Burgers equation:

$$\frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}. \quad (1.3.8)$$

The Burgers equation refers to a parabolic type and is a one-dimensional case of the Navier-Stokes equations. The Navier-Stokes equations describe two-dimensional wave processes in a viscous incompressible fluid:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \nabla) \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + (\zeta + \frac{\eta}{3}) \nabla \nabla \mathbf{v}. \quad (1.3.9)$$

The process of propagation of a bounded sound beam in a nonlinear medium is described by the two-dimensional Khokhlov-Zabolotskaya-Kuznetsov equation:

$$\frac{\partial}{\partial \tau} \left[ \frac{\partial p'}{\partial z} - \frac{\varepsilon}{c_0^3 \rho_0} \frac{\partial p'}{\partial \tau} - \frac{b}{2c_0^3 \rho_0} \frac{\partial^2 p'}{\partial t^2} \right] = \frac{c_0}{2} \Delta_{\perp} p', \quad (1.3.10)$$

where, $\Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian in transverse coordinates.

In the theory of vibrations and solitons applicable equation to the Klein-Gordon:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = V'(u), \quad (1.3.11)$$

here $V'(u)$ is a certain nonlinear function, depending on $u$. When $V'(u) = \sin u$ the Klein-Gordon equation is transformed into the so-called sin-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin u, \quad (1.3.12)$$

The sin-Gordon equation is used to describe topological solitons with the geometry of surfaces of negative Gaussian curvature.

The theory of nonlinear waves also uses an equation having a solution of the type of running waves of arbitrary shape.

$$\left( 1 - \frac{\partial u^2}{\partial t} \right) \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x \partial t} - \left( 1 + \frac{\partial u^2}{\partial x} \right) \frac{\partial^2 u}{\partial t^2} = 0. \quad (1.3.13)$$
This is the *Born-Infeld* equation; it describes the phase discontinuities in the interaction of two solitons.

### 1.4 Methods for Solving Nonlinear Equations

#### The Method of Successive Approximations

There are various approximate analytical methods for solving problems in mathematical physics. The choice of a suitable method for solving equations depends on the nature of the problem under consideration. *Nonlinear equations* can be divided into two classes – algebraic and transcendental. Algebraic equations are equations that contain only algebraic functions (integer, rational, irrational). Equations containing other functions (trigonometric, exponential, logarithmic, and others) are called transcendental [Amosov, 1994], [Shup Terry, 1990].

Methods for solving nonlinear equations can be divided into two groups:

1. *exact methods*;
2. *iterative methods*.

Exact methods allow us to write the roots in the form of a finite relation (formula). As is known, many equations and systems of equations do not have analytic solutions. First of all, this applies to most transcendental equations. In some cases, the equation contains coefficients known only approximately. To solve them, iterative methods are used with a given degree of accuracy.

Solving the equation by the *iterative method* means: to establish whether it has roots, how many roots, and to find the values of the roots with the required accuracy. Approximate values of the roots (initial approximations) can also be known from the physical meaning of the problem, from solving a similar problem with other input data, or can be found *graphically*.

The iterative process consists in the successive refinement of the initial approximation \( x_0 \). Each such step is called an iteration. As a result of iterations there is a sequence of approximate values of the root \( x_1, x_2, ..., x_n \). If these values increase with the number of iterations \( n \) approach the true value of the root, then it is said that the iterative process converges. Iterative methods include the method of *successive approximations* (or the simple iteration method).
The essence of the method of successive approximations is as follows. The equation \( f(x) = 0 \), to be solved is rewritten in the form

\[
x = \phi(x)
\]  

(1.4.1)

After this, the initial approximation \( x_1 \) is chosen and is substituted into the right-hand side of equation (1.4.1). The obtained value \( x_2 = \phi(x_1) \) is taken as the second approximation for the root. If an approximation \( x_n \) is found, then the following approximation \( x_{n+1} \) is determined by the formula

\[
x_{n+1} = \phi(x_n).
\]

Suppose that after a few approximations we find that a given degree of accuracy the equality \( x_n \approx x_{n+1} \). Since \( x_{n+1} = \phi(x_n) \), then this means that the equation is satisfied with a given accuracy \( x_n \approx \phi(x_n) \), that is, \( x_n \) which is the approximate value of the root of the equation \( x = \phi(x) \).

When using the method of successive approximations, it is necessary to find out the following features:

1. if there is equality \( \lim_{n \to \infty} x_n = \xi \), then whether the number \( \xi \) solution of equation \( x = \phi(x) \) ?
2. always whether the sequence \( x_1, \ldots, x_n \) converges to some number \( \xi \)?
3. how fast numbers approach \( x_1, \ldots, x_n \) to the root \( \xi \) equations \( x = \phi(x) \) ?

Answering the first question, let’s say, that the numbers \( x_1, \ldots, x_n \) approaching the number \( \xi \). Consider the equality \( x_{n+1} = \phi(x_n) \), Giving an expression of the next approximation in the previous. With increasing \( n \), its left-hand side approaches to \( \xi \), and the right-hand side to \( \phi(\xi) \). Therefore in the limit we obtain \( \xi = \phi(\xi) \), that is, it \( \xi \) is the root of equation \( x = \phi(x) \).

The second question is ambiguous; for this we consider the geometric representation of the method of successive approximations [Vilenkin, 1968].

**Geometric Interpretation of the Method of Successive Approximations**

Finding the root \( \xi \) of the equation \( x = \phi(x) \) is nothing more than finding the abscissa of the point \( A \) of the intersection of the curve \( y = \phi(x) \) with the line \( y = x \). We construct \( xOy \) the graphs of the functions \( y = x \) and \( y = \phi(x) \). Each real root \( \xi \) of equation (1.4.1) is the abscissa of the
intersection point $A$ of the curve $y = \phi(x)$ with the straight line $y = x$ (Figure 1.4.1a).

Starting from some point $A_0(x_0, \phi(x_0))$, we build a broken line $A_0B_1A_1B_2A_2...$ ("stairs"), the links of which are alternately parallel to the axis $Ox$ and axis $Oy$, vertices $A_0, A_1, A_2, ...$ on the curve $y = \phi(x)$, and the vertices $B_1, B_2, B_3, ...$ on the line $y = x$. Common abscissae of the points $A_1$ and $B_1, A_2$ and $B_2, ...$ obviously represent respectively the successive approximations $x_1, x_2, ...$ root $\xi$.

Another kind of broken line is also possible $A_0B_1A_1B_2A_2...$ — a «spiral» (Figure 1.4.1b). The solution in the form of a «stairs» is obtained if the derivative $\phi'(x)$ is positive, and the solution in the form of a «spiral», if $\phi'(x)$ negative.

Thus, the geometrical meaning of the method of successive approximations is that we move to the desired point of intersection of the curve and the straight line along the broken line. In this case, the vertices of the broken line lie alternately on the curve and on the straight line, and the sides alternately have horizontal and vertical directions (Figure 1.4.1).

In Figure 1.4.1a, b the curve $y = \phi(x)$ in the neighborhood of the root $\xi$ — that is, $|\phi'(x)| < 1$, and the iteration process converges. However, if we consider the case when $|\phi'(x)| > 1$, then the iteration process can be divergent (Figure 1.4.2). Therefore, for practical application of the method of successive approximations, it is necessary to find out sufficient conditions for the convergence of the iterative process.

### 1.5 The Basic Laws of Hydrodynamics of an Ideal Fluid

Hydrodynamics studies the motion of liquids and gases; the basic principles of hydrodynamics were established by Euler, Bernoulli and Lagrange.
To derive the basic laws, we shall first consider the fluid to be ideal, that is, the liquid has no internal friction, and the mechanical energy does not go over into thermal energy. We also neglect heat exchange between different volumes of the liquid. This means that all processes occur at constant entropy, and the stressed state of the liquid is characterized by a single scalar quantity, the pressure $p$.

The motion of a fluid can be considered definite if all the quantities characterizing the fluid (the particle velocity $v$, pressure $p$, density $\rho$, temperature $T$ etc.), are given as functions of the coordinates and time. This method of specifying fluid motion was proposed by Euler. At the same time, fixing a certain point of space, we follow the change in time of the corresponding quantities at this point. Fixing the moment of time, we find the change of this quantity from point to point. However, no information on what kind of a particle of liquid is at a given point at a given time and how it moves in space, we do not directly have.

Another way of describing the flows, based on the description of the motion of individual liquid particles, was proposed by Lagrange. In the Lagrangian description, attention is fixed to certain particles of the liquid and can be traced, as a change over time of their position, speed, pressure, density, temperature and other quantities in their environment.

These two descriptions of the movement are completely equal, and the choice of one of them in each particular case is dictated only by considerations of convenience. Most instruments measure fluid characteristics at a fixed point, that is, they give Euler information. If you paint (mark) a part of the liquid, then by spreading the paint, Lagrangian motion information is obtained. Therefore, the Lagrange method is simpler; it describes the diffusion process associated with the movement of the particles.

Figure 1.4.2 The divergent iterative process.
The System of Equations of Hydrodynamics

**The continuity equation.** One of the fundamental equations of hydrodynamics is the equation of continuity, or the law of conservation of matter. It shows that the mass of a liquid in a volume that covers the same particles all the time is preserved. Mathematically, this can be written as follows [Brekhovskikh, 1982]:

\[
\frac{d}{dt} \int_V \rho dV = 0 \tag{1.5.1}
\]

or in partial derivatives

\[
\int_V \left( \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \mathbf{v} \right) dV = 0 \tag{1.5.2}
\]

whence, by virtue of the arbitrariness of the volume \( V \), equating the integrand with zero, we obtain the required equation of continuity

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \mathbf{v} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0, \quad \text{or} \quad \frac{d}{dt} \rho + \rho \nabla \mathbf{v} = 0 \tag{1.5.3}
\]

The integral form of the continuity equation (1.5.2) has a simple physical meaning (with the Ostrogradskii-Gauss theorem)

\[
\frac{\partial}{\partial t} \int_V \rho dV = -\int_s \rho \mathbf{v} n dS \tag{1.5.4}
\]

– the speed change of the mass of the liquid inside a fixed volume \( V \) is equal to the mass of the liquid (with the opposite sign) flowing out of the volume through its surface per unit time \( S \).

**The Euler equation.** We now turn to the derivation of the equation of fluid motion, also called the Euler equation. For this it is sufficient to apply Newton's second law: the time derivative of the momentum of a certain volume of liquid is equal to the sum of the forces acting on this volume:

\[
\frac{d}{dt} \int_V \rho \mathbf{v} dV = \mathbf{F} + \mathbf{F}_s \tag{1.5.5}
\]
where
\[ F = \int V \rho f dV \]
is the external volume force \((f - \text{is the force per unit mass})\); \(F_s\) – is the force acting on the volume \(V\) from the environment side through the bounding surface \(S\),
\[ F_s = -\int_S \rho n dS, \]
where \(n\) – is the outer normal to \(S\).

Using the arbitrariness of the volume \(V\), we obtain the desired Euler equation
\[
\frac{dv}{dt} = \frac{\partial v}{\partial t} + (\nabla \nabla) v = -\frac{\nabla p}{\rho} + f \quad (1.5.6)
\]

Completeness of the system of equations. Equation (1.5.6) together with the continuity equation (1.5.3) is four scalar equations for five scalar quantities (density \(\rho\), pressure \(p\) and three components of the velocity vector \(v_i\)). Consequently, the system of equations is not yet closed. The thermodynamic equation of state connecting the three quantities: pressure \(p\), density \(\rho\) and entropy \(s\)
\[ p = p(\rho, s), \quad (1.5.7) \]
and the equation for entropy. As we noted, the entropy of a given liquid particle remains constant, i.e.
\[ \frac{ds}{dt} = \frac{\partial s}{\partial t} + v \nabla s = 0 \quad (1.5.8) \]

Entropy \(s\) can be excluded from the equations, for this we need to take from (1.5.7) take the time derivative and denote by
\[ c^2 = \frac{\partial p}{\partial \rho}, \quad (1.5.9) \]

As a result, we obtain
\[
\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \quad (1.5.10)
\]
where \( c^2 \) should already be considered a given function \( p \) and \( \rho \), and is the speed of sound in the medium.

If the liquid is barotropic, i.e. the pressure depends only on the density

\[
p = p(\rho)
\]  

(1.5.11)

To then the last equation will be the closure for the system (1.5.3) and (1.5.6). The form of the function \( p(\rho) \) depends on the properties of the liquid or gas under consideration.

A complete closed system of equations (1.5.3), (1.5.6) and (1.5.10) is called the system of hydrodynamics equations for an ideal fluid:

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \nabla \rho + \rho \nabla \mathbf{v} = 0 \quad \text{the continuity equation,}
\]

\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{f} \quad \text{the Euler equation (motion),}
\]

\[
\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \quad \text{equation of state.}
\]

To solve specific problems, the corresponding boundary conditions must be added to the equations. An example of the force \( \mathbf{f} \), entering the Euler equation (1.5.6) is the force of gravity. In this case \( \mathbf{f} \) is equal to the vector \( \mathbf{g} (|\mathbf{g}| = 9.81 \text{ m/s}^2) \), directed to the center of the Earth. The equations can also include the forces of attraction of the moon and the sun, which are taken into account in the theory of the sea tides. In the dynamics of the ocean and the atmosphere, the non-inertiality of the reference system associated with the rotating Earth is essential. In this case inertia forces must be introduced into the system of equations of hydrodynamics:

\[
\text{centrifugal } \mathbf{f}_{\text{ctr}} = \Omega^2 r = \nabla(\Omega^2 r^2/2),
\]

where \( \Omega \) – is the Earth’s rotation frequency, and the Coriolis force \( \mathbf{f}_c = -2\Omega \times \mathbf{v} \). We note that all the forces considered by us, other than the Coriolis force, are potential, that is, representable in the form of a gradient from the potential function: \( \mathbf{f} = -\nabla \mathbf{u} \).

In connection with the assumption of incompressibility of a fluid, the density of each particle must remain constant:

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \nabla \rho = 0.
\]
It does not follow from the equation of state (1.5.10) that $\frac{dp}{dt} = 0$, since in an incompressible fluid the speed of sound $c$ is infinitely large. Thus, in an incompressible fluid equation (1.5.10) is replaced by $d\rho / dt = 0$ and, taking into account the continuity equation, we have

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \nabla \rho = 0, \quad \nabla \mathbf{v} = 0,$$

thus the system of hydrodynamic equations remains closed.

### 1.6 Linear Equations of Hydrodynamic Waves

Due to wave movements carried out in a fluid energy transfer over large distances without mass transfer. The various forces acting on liquid particles during their motion lead to different wave motions. Let us consider the main types of waves in a liquid in the so-called linear approximation, when the waves propagate independently of each other. Gravity surface waves and internal waves in the thickness of a liquid are related to waves due to the action of gravity. As a result of the compressibility of the medium, acoustic waves propagate in it.

**Linearization of the equations of hydrodynamics.** When studying wave processes in a liquid, we must start from the basic equations of hydrodynamics ((1.5.3), (1.5.6) and (1.5.10)) [Brekhovskikh, 1982]:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} = - \frac{\nabla p}{\rho} - g \nabla z - 2\Omega \times \mathbf{v},$$

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0,$$

$$\frac{d\rho}{dt} = c^2 \frac{d\rho}{dt}, \quad c^2 = \left( \frac{\partial p}{\partial \rho} \right),$$

Here, in Euler’s equation, we included the force of gravity directed vertically downward, and the Coriolis force $-2\Omega \times \mathbf{v}$, arising in a fluid rotating with frequency $\Omega$ (for example, on a rotating Earth). Let us consider the theory of wave propagation in an ideal fluid, without touching on the questions of their generation.

An important circumstance in the study of wave motions in a liquid is the nonlinearity of the hydrodynamic equations (1.6.1), so the exact theory