Granular Geomaterials Dissipative Mechanics

Theory and Applications in Civil Engineering

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Preface

Granular materials are present in numerous sectors of economic activity outside civil engineering, from agriculture and agro-industry to pharmaceutical and chemical industries, mining industry, etc. It is estimated that more than two-thirds of raw materials used by world industries are in the form of granular materials, involving gigantic quantities, about 10 billion tons each year, of which processing and transport represent about 10% of energy consumption worldwide [DUR 96]. However, most often, the methods for their process remain rather traditional and lack optimization.

Regarding geomaterials, sand for the construction industry is the second most consumed natural resource after water [LEH 018], and its extraction represents serious environmental issues in certain areas, (including the disappearance of beaches and retreat of shoreline).

Construction of large civil engineering infrastructures commonly involves large volumes of earthfills and rockfills, constituted by sand, gravel, and rock blocks, sometimes up to tens of millions of cubic meters or even more, as in highways or railway platforms, marine infrastructures or large rockfill dams (see Figure 1). Examples of these include the Grand-Maison Dam in France (height 160 m, volume 14 hm$^3$) with a central compacted clay core, or the Campos Novos Dam in Brazil (202 m, 13 hm$^3$) with an impervious concrete slab on the upstream face, which will be discussed in Chapter 10.

For this last type of dams, which has become dominant in dam construction today, a major part of the design methods is based on the empirical extrapolation of the standard ones used (in the past) for lower dams. This empirical approach, based on experience, has led to serious technical accidents during commissioning on very high dams in the mid-2000s. As a consequence, concern in the profession has arisen, prompting a return to more rational approaches in design, and particularly
engineering approaches, through structural analysis and relevant material testing as should be the case for any large civil engineering structure. This highlights the need to improve our knowledge of the behavior of the granular geomaterials constituting these infrastructures, as well as of the behavior of these large structures. A way for such improvement may be sought in the integration of physical local phenomena within the materials, up to the scale of the engineering structures.

Figure 1. Large earth and rockfill infrastructures in civil engineering. (a) High-speed railway infrastructures. (b) Marine works. (c) Rockfill dams (Grand-Maison Dam – photo EDF). For a color version of the figure, please see www.iste.co.uk/frossard/geomaterials.zip

This book, resulting from a long-term work into the physics of granular materials as well as engineering of large civil works, is an attempt to relevantly move forward proposing a new vision of mechanical behavior of these granular geomaterials, through an original dissipative approach.

After an introductory section on background and key assumptions, the book begins on the main theoretical features of dissipative structures induced by elementary contact friction associated with specific statistical mechanics properties within granular materials in slow motion, and their multi-scale expression into key tensor relations, Chapters 1 and 2.

These dissipation relations and related features constitute the backbone of practical applications developed further in this book, starting in Chapter 3 focusing
on strain localization and shear band detailed features, leading to the process of failure lines generation.

Then, Chapters 4–8 develop practical applications of the main macroscopic energy-dissipation equation and related features to a large set of key properties of great relevance in geotechnical and civil engineering, mainly:

– the failure criterion, resolving into the Coulomb Criterion under critical state;

– the relationships between shear strength and volume changes, expressed in generalized 3D stress–dilatancy relations, resolving into classical Rowe’s relations in particular conditions;

– the characteristic state;

– cyclic compaction features under alternate shear movements;

– the geostatic equilibrium \( (K_0) \), achieving a relation close to the Jaky formula.

Chapter 6 is focused on a wide set of experimental data collected worldwide, covering most of the experimental apparatuses, which thoroughly validate the dissipative approach of the mechanical behavior.

Although a major part of the book is focused on features induced by contact friction, the last part, Chapters 9 and 10, presents the key results on practical features resulting from particle breakage, the other main dissipative process after contact friction. These results include explicit incidences of size effects in shear strength, slope stability and safety factors, deformations and settlements in rockfill embankment dams.

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I.1. Background

Since Coulomb’s historical publication of 1773 [COU 73], many investigations have been carried out on the role of physical friction in the mechanical behavior of granular materials. These investigations, supported by pioneering works published by Rankine [RAN 57], Prandtl [PRA 20], Caquot [CAQ 34], Terzaghi [TER 43], and many others, have been progressively incorporated into the body of knowledge of Soil Mechanics, a pillar of civil engineering sciences. However, a direct link between the initial cause – friction at the grain contacts – and the elements of practical interest concerning the behavior at a macroscopic scale, such as the failure criterion or the 3D stress–strain relationship, has not been clearly established. Significant advances in this direction have been made, such as Rowe’s stress–dilatancy theory [ROW 62], which was enriched later by Horne in 1965–1969 [HOR 65, HOR 69], or more recent statistical mechanics approaches. Their conditions of validity, however, limited to axisymmetric stress conditions, or 2D granular assemblies made of disks, are more restrictive to apply them in a general case.

The approach presented here has a larger scope and finds a solution to more general 3D quasi-static problems for granular media with grains of random irregular shapes (Figure I.1). It provides us with the access to an explicit expression of a wide set of macroscopic properties such as stress–dilatancy laws, failure criterion, strain localization with internal structure of the shear bands, orientation and development of failure lines, the intricate relations between friction, shear strength, and volume changes, and the cyclic compaction under alternate shear motion.

This specific multi-scale approach was developed from the following observations:
– Granular materials, even considered as pseudo-continuum at large scale, remain densely discontinuous at small scale; therefore, the large-scale pseudo-continuous behavior is likely to be highly conditional upon the small-scale behavior of elementary discontinuities: the inter-granular contacts.

– Within a granular material in motion, internal mechanical processes are highly irreversible, and the main source of this irreversibility is at small-scale dissipation of energy by sliding friction at inter-particle contacts.

Figure I.1. Typical rockfill (basalt) used in civil engineering. For a color version of the figure, please see www.iste.co.uk/frossard/geomaterials.zip

– This small-scale energy dissipation by contact friction can be simply formulated with relevant local elementary quantities, such as inter-granular contact forces and contact sliding movements, by direct application of classical friction laws.

– By a multi-scale analysis, the transposition to the macroscopic scale should lead to a macroscopic energy dissipation relation, linking macroscopic relevant quantities, such as stress and strain rates, and connecting to the thermodynamics of dissipative processes.
In classical standard mechanical behaviors, such as basic fluid mechanics in hydraulics or standard elastoplasticity, the energy dissipation may often be conceptually regarded as a perturbation or a complement within the main framework provided by a regular non-dissipative behavior (e.g. the “perfect incompressible fluid” mechanics in hydraulics, or elasticity in elastoplasticity).

In most of the chapters in this book, energy dissipation by contact friction will stand “alone on stage”; therefore, all of the properties developed are its direct consequences: the whole set of behavioral characteristics displayed appear as a mechanical dissipative structure, hence the name dissipative mechanics.

I.2. Main assumptions

To achieve a clear formulation, this energy dissipation approach requires a set of material and mechanical assumptions, selected to preserve the core of the mechanical behavior. The granular media considered are under slow motion, slow enough to neglect macroscopic dynamical effects or variations in kinetic energy (quasi-static conditions). These media are material sets constituted by rigid, cohesionless mineral particles, with random irregular convex shapes, resulting in no resistance to macroscopic tensile stresses. The inter-particle contacts are unilateral and purely frictional with a uniform friction coefficient.

Relevant internal movements considered in the granular media in motion are the relative sliding movements at contacts. Particle rotations do exist in the granular mass in movements, but remain limited to kinematic shear rotations on average (i.e. with random irregular shapes, there are no macroscopic significant “ball-bearing-like” movements within the granular mass in motion, as described in Chapters 1 and 6). Therefore, the incidence of macroscopic strains of rolling and spinning relative movements is considered here on average as relatively negligible to sliding movements.

Relevant internal forces considered in the granular media in motion are locally the resultant vectors of contact forces exerted on very small contact areas, which are considered as point contacts; the energy effects of contact moments (rolling and spinning) are considered here on average as relatively negligible to the effects of resultant vectors. In this condition, the internal work is made only by contact forces against the relative contact displacements, and the mechanical energy dissipated in the contacts is due to contact sliding motions.

If the granular material is saturated by a fluid filling the inter-granular voids, the fluid pressure is taken as the origin of pressures: the reasoning is conducted on inter-granular forces or macroscopic effective stresses.
With the Eulerian description of the equivalent pseudo-continuum, compressive stress and contraction strain will be denoted as positive, according to the usual conventions in geomechanics. The local values of these stress and strain rates will be considered as the sum of:

- an average component, on which the large-scale gradients are exerted due to external actions (such as gravity);
- a component of local random fluctuations, due to the inherent heterogeneity of the medium.

Under regular boundary conditions, the correlations between these fluctuations will be considered to decay sufficiently with the distance beyond a certain scale, so that they have a negligible effect on the macroscopic work rate of internal forces and on the norm of internal actions.

The granular mass in slow dissipative motion close to static equilibrium may be considered resulting from a statistical population of dissipative moving contacts with greater degrees of freedom. Therefore, we assume that it satisfies a “minimum dissipation rule” stated as follows: under regular, monotonic, quasi-equilibrium boundary conditions, the moving medium tends toward a regime of minimum energy dissipation compatible with the imposed boundary conditions; this regime is independent of the initial particular conditions. This rule, strongly suggested by a set of theoretical and experimental results, may be shown [FRO 04] to be a corollary of the Prigogine minimum entropy production theorem based on the thermodynamics of dissipative systems near equilibrium [PRI 68], see Appendix A.I.1.

I.3. Key of the multi-scale approach: the internal actions, a new tensor concept

Deriving constitutive relations from a local discontinuous granular media toward its equivalent pseudo-continuum representation raises numerous basic questions of mechanics, which bring up the need for some new “tool”, both conceptually relevant and clearly formalized, involving the following six key properties regarding the mechanics:

- to be a simple function of internal movements and internal forces, including a built-in orientation referential objectively linked to the material set in motion;
- to be an additive physical quantity: the quantity over a whole material set shall be the sum of the quantities related to parts of the whole set (eventually with the addition of boundary terms), which is not the case for internal movements or internal forces considered separately;
– to have a physical meaning in the discontinuous media, both at local elementary scale (the particle) and the global scale (set of particles in contact), in order to derive relations between local properties (local scale) and average properties (global scale);

– to also have physical meaning in the equivalent pseudo-continuum, in order to allow the transposition of properties derived in the discontinuous media toward its equivalent continuum representation;

– to be compatible with the mechanical heterogeneity, inherent to granular media (strongly heterogeneous distributions of internal movements and internal forces);

– to have a direct link with strain energy, or more precisely, the work rate of internal forces, in order to provide a simple formulation of energy balance, interchanges within the material involved in the energy dissipation.

Such a tool with these six properties has been found in the second-order symmetric tensors resulting from the symmetric product of internal forces and internal movements, holding the work rate of internal forces as the first invariant.

This tool revealed the tensor structures induced by contact friction (Chapter 1) and made possible the general multi-scale approach from an elementary contact to the macroscopic behavior presented in the following chapters. It turned out to be particularly relevant for our specific approach of contact friction dissipative structure, as the resulting key behavior equations operates on its eigenvalues.

From the author’s point of view, the above considerations justify paying particular attention to this new tool and proposing a specific name: the internal actions.
The notations in this book have been kept consistent with the author’s previous publications on the subject, except for the physical contact friction between mineral particles (previously denoted as $\psi$), here indicated as $\phi_\mu$, in order to better correlate Chapters 4 and 5 with classical developments made in the UK in the 1960s.

**Discontinuous granular medium**

**Elementary contact**

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{v}(a/b)$</td>
<td>Relative sliding velocity at the contact between rigid particles $a$ and $b$</td>
</tr>
<tr>
<td>$\mathbf{f}(a/b)$</td>
<td>Resultant contact force exerted by particle $a$ on particle $b$</td>
</tr>
<tr>
<td>$\phi_\mu$</td>
<td>Physical friction at the contact between mineral particles</td>
</tr>
<tr>
<td>$\dot{W}$</td>
<td>Work rate of internal contact forces (here equal to energy rate dissipated by friction, as the energy storage is neglected)</td>
</tr>
<tr>
<td>$\mathbf{p}(a/b)$ or $\mathbf{p}(c)$</td>
<td>“Tensor of internal actions for an elementary contact”, resulting from the symmetric tensor product of the vectors $\mathbf{v}(a/b)$ and $\mathbf{f}(a/b)$</td>
</tr>
<tr>
<td>$p_i(c)$</td>
<td>Eigenvalue of tensor $\mathbf{p}(c)$</td>
</tr>
<tr>
<td>$p^+(c)$</td>
<td>“Input power” at elementary contact $c$, the sum of positive eigenvalues of tensor $\mathbf{p}(c)$</td>
</tr>
<tr>
<td>$p^-(c)$</td>
<td>“Output power” at elementary contact $c$, the sum of negative eigenvalues of tensor $\mathbf{p}(c)$</td>
</tr>
</tbody>
</table>
$N\{p\}$ Octahedral norm of tensor $p(c)$, the sum of absolute values of its eigenvalues

**Granular mass**

$P(A)$ “Tensor of internal actions in the granular mass $A$”, resulting from the sum of all the tensors of internal actions for elementary contacts $p(a/b)$ or $p(c)$, included in the granular mass

$P^+(A)$ and $P^-(A)$ Input power and output power in the granular mass, the sum of positive eigenvalues (and of negative eigenvalues, respectively) of $P(A)$

$R(A)$ “Internal feedback rate function” (population effect within the granular mass $A$), related to the degree of disorder within the statistical distribution of the moving contact orientations

$\phi^*_\mu$ Apparent inter-granular friction, including the effect of both mineral contact friction $\phi_\mu$ and population effect $R(A)$ (in this book, numerical examples and figures: $\phi^*_\mu = 30^\circ$ except otherwise stated)

$S$ Abbreviated notation for sin $\phi^*_\mu$ in complex expressions

**Equivalent pseudo-continuum**

$\pi, \pi_i$ “Tensor of internal actions for equivalent pseudo-continuum”, defined as the tensor contracted symmetric product between stress tensor (internal forces) and strain rate tensor (internal movements), and its eigenvalues

$\pi^+$ and $\pi^-$ Input power and output power in the equivalent pseudo-continuum, the sum of positive eigenvalues (and of negative eigenvalues, respectively) of the tensor $\pi$

$\sigma, \sigma_i$ Macroscopic Eulerian (Cauchy) stress tensor, and principal stress. Compressive stresses considered as positive; by convention, all stresses considered are effective stresses (so, the $'$ is omitted)

$\tau, \sigma_n$ Shear stress, normal stress

$\bar{\sigma}, \hat{\sigma}$ Average value over a domain, and local fluctuations relative to this average value, for tensor $\sigma$

$\mathcal{D}$ Material domain in motion
Specific notations for plane strain situations: shear stress and the half sum of major and minor principal stresses

Macroscopic Eulerian strain rate tensor, principal strain rate, volume strain rate, shear strain rate (contraction strains considered as positive by convention)

Scalar parameter defining the deviatoric stress state: 

Scalar parameter defining the deviatoric strain rate state:

Generalized dilatancy rate, the scalar function of the strain rate tensor:

Lode angle for the deviatoric stress state

Angular position for the deviatoric strain rate state, within principal stresses referential (coaxial situations)

Norm of the tensor , the sum of absolute values of eigenvalues

“Internal friction,” defined by under monotonous shear solicitation

Specific volume

Work rate of internal forces per unit volume (here, fully dissipated)

Specific dissipation rate, per unit mass (here, )

Specific deformation in a material domain, defined by a functional over time of the strain rate tensor
**Rockfill dams, scale effects, stability**

\( b \)  
Material parameter, the exponent in parabolic shear strength envelope

\( m \)  
Material parameter, the exponent in Weibull’s statistical distribution of mineral particles crushing strength

\( F_s \)  
Safety factor against shear failure, defined as the ratio between mobilizable shear strength resistance and exerted shear stresses at equilibrium
Fundamentals: The Tensor Structures Induced by Contact Friction

This chapter details the tensor structures induced by contact friction, whose prominent characteristics are summarized in the synoptic Figure 1.1 – from the scale of an elementary contact to the scale of macroscopic equivalent pseudo-continuum – displaying how energy dissipation by contact friction induces the structures in the eigenvalues of internal actions at all scales.

These structures are shown to result in energy dissipation equations operating on internal action invariants, at every scale, integrating population effects from the mesoscopic scale to the macroscopic one: the “internal feedback” effect resulting from interactions between adjacent inter-granular contacts in motion, which is a kind of micro-mechanical mixed arching and domino effect.

These structures are shown to result from energy dissipation by contact friction associated with the “minimum dissipation rule” detailed in the Introduction to this book. At the mesoscopic scale, the minimum dissipation solutions, i.e. the distributions of elementary contact actions achieving the minimum dissipation, are shown, in general, to present high polarization of internal contact action orientations. Under plane strain conditions, the mesoscopic minimum dissipation solution results in the polarization of elementary contact sliding motion corresponding to Rankine’s slip lines.

The last part of this chapter is focused on the correspondence between the discontinuous granular mass (mesoscopic scale) and its equivalent pseudo-continuum (macroscopic scale), leading to the macroscopic equation of energy dissipation by contact friction near minimum energy dissipation.
Figure 1.1. Synopsis of multiscale tensor structures induced by contact friction.
For a color version of the figure, please see www.iste.co.uk/frossard/geomaterials.zip
From the author’s point of view, these tensor structures and their material expressions in the polarized distributions of internal actions can be seen as dissipative structures induced by a specific form of energy dissipation by contact friction.

1.1. Microscopic scale: the elementary inter-granular contact

1.1.1. Vector formulation of energy dissipation

Consider a simple contact \( c \) between two grains \( a \) and \( b \), sliding with a relative velocity \( \mathbf{v}(a/b) \) under a contact force \( \mathbf{f}(a/b) \), with an elementary friction angle at contact \( \mu \) (Figure 1.1). The elementary laws of friction result in the following relation between the two vectors:

\[
\mathbf{f}(a/b) \cdot \mathbf{v}(a/b) = \sin \mu \cdot \| \mathbf{f}(a/b) \| \cdot \| \mathbf{v}(a/b) \|.
\]  

[1.1]

It may be noted that the above-mentioned vector equation still holds even when the movement stops (i.e. \( \mathbf{v}(a/b) \) becomes null) or when the contact disappears as the grains separate in the motion (i.e. \( \mathbf{f}(a/b) \) becomes null). Equation [1.1] expresses the equality between the work rate of contact forces on its left-hand side, and an always positive function – then a dissipation function – on its right-hand side. The vector equation then corresponds to the energy dissipation during sliding.

1.1.2. Tensor formulation of energy dissipation

These two vectors may be considered as the internal movement and internal force of our contact \( c \). From their symmetrical product, the “tensor of elementary contact actions” \( \mathbf{p}(c) \), a symmetrical second-order tensor, whose trace is the mechanical work rate produced by the contact force \( \mathbf{f}(a/b) \) during sliding, can be defined as follows:

\[
\mathbf{p}(c) = \frac{1}{2} \left[ \mathbf{f}(a/b) \otimes \mathbf{v}(a/b) + \mathbf{v}(a/b) \otimes \mathbf{f}(a/b) \right]
\]

or in components

\[
p_{ij} = \frac{1}{2} \left[ f_i v_j + v_i f_j \right]
\]

then

\[
Tr \{ \mathbf{p}(c) \} = \mathbf{f}(a/b) \cdot \mathbf{v}(a/b).
\]  

[1.2]

It may be noted that, by its definition, this tensor is also independent of the order affected by the considered grains or particles, either the contact of grain \( a \) on grain \( b \)
(earlier denoted as \(a/b\)) or the reverse, \(b/a\), because the relative velocity and exerted force in the \(b/a\) case are opposite to the ones in the case \(a/b\). This justifies the notation \(p(c)\) that now relates this variable to the contact \(c\), independent of the way we consider it, either contact of grain \(a\) on grain \(b\) or the reverse \(b/a\).

This tensor of elementary contact actions \(p(c)\) can be easily diagonalized in its natural basis formed by the two bisecting lines in the directions of \(v(a/b)\) and \(f(a/b)\) (eigendirections numbered 1 and 3), and their common normal (eigendirection numbered 2). In this natural basis

\[
p(c) = \|f(a/b)\| \cdot \|v(a/b)\| \begin{bmatrix}
\cos^2 \left(\frac{\pi}{4} - \frac{\phi}{2}\right) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\sin^2 \left(\frac{\pi}{4} - \frac{\phi}{2}\right)
\end{bmatrix}.
\]

From the three eigenvalues of \(p(c)\), we can define the symmetrical function as follows:

\[
N\{p(c)\} = |p_1(c)| + |p_2(c)| + |p_3(c)|.
\]

This function, which is a tensor norm of \(p(c)^1\), named “octahedral norm” in the following, is related to the Euclidian norms of the two vectors \(v(a/b)\) and \(f(a/b)\) by the following relation, resulting from the diagonalized expression [1.3]:

\[
N\{p(c)\} = \|f(a/b)\| \cdot \|v(a/b)\|.
\]

Merging equations [1.1], [1.2], and [1.5], we can now express the dissipation relation resulting from the elementary laws of friction by a relation between the eigenvalues of \(p(c)\), which corresponds to the tensor equation of the energy dissipation by friction at a single contact point.

\(^1\) This norm, also known as “Manhattan or Taxicab norm”, belongs to the mathematical family of \(p\)-norms including also the Euclidean norm and the Supremum norm. However, unlike the Euclidean norm, our octahedral norm is a piece-wise linear function, each linearity domain corresponds to one face of its unit ball, being a regular octahedron. This piece-wise linearity will turn out to be a key property when dealing with the pseudo-continuum heterogeneous mechanical behavior like shear banding (Chapters 2 and 3).