Fractional Brownian Motion
Chapter 2. Distance Between fBm and Subclasses of Gaussian Martingales

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<td>$A^\top$</td>
<td>The transpose of the matrix $A$</td>
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<td>$A^*$</td>
<td>The conjugate transpose (Hermitian transpose) of the matrix $A$</td>
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<tr>
<td>$a_i\cdot$</td>
<td>The transpose of the $i$th row of the matrix $A = (a_{ij})$</td>
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<td>$B(\alpha, \beta)$</td>
<td>The beta function</td>
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<td>$B^H$</td>
<td>Fractional Brownian motion with Hurst parameter $H$</td>
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<td>$\mathcal{B}(\mathbb{R})$</td>
<td>Borel $\sigma$-algebra on $\mathbb{R}$</td>
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<td>$C^\lambda([a, b])$</td>
<td>Space of Hölder continuous functions $f: [a, b] \to \mathbb{R}$ with Hölder exponent $\lambda \in (0, 1]$ equipped with the norm $|f|<em>\lambda = \sup</em>{t \in [a, b]}</td>
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<td>$D^\alpha_{a^+} f$</td>
<td>Riemann–Liouville left-sided fractional derivative of order $\alpha$</td>
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<td>Riemann–Liouville right-sided fractional derivative of order $\alpha$</td>
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<td>$f_{a^+}(x)$</td>
<td>$= (f(x) - f(a^+)) \mathbf{1}_{(a, b)}(x)$</td>
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<td>$T^\alpha_{a^+} f$</td>
<td>Riemann–Liouville left-sided fractional integral of order $\alpha$</td>
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\( \mathcal{I}_b^\alpha f \)  
Riemann–Liouville right-sided fractional integral of order \( \alpha \)

\( L_p([a,b]) \)  
Space of measurable \( p \)-integrable functions \( f : [a, b] \rightarrow \mathbb{R}, \ p > 0, \) equipped with the norm

\[
\|f\|_{L_p([0,T])} = \left( \int_0^T |f(s)|^p \ ds \right)^{\frac{1}{p}}
\]

\( L_H^2([0,T]) \)  
Space of functions \( f : [0, T] \rightarrow \mathbb{R} \) such that

\[
\int_0^T \int_0^T |f(s)||f(u)||u-s|^{2H-2} \ du \ ds < \infty
\]
equipped with the norm

\[
\|f\|_{L_H^2([0,T])} = \left( H(2H - 1) \right. \\
\times \left. \int_0^T \int_0^T |f(s)||f(u)||u-s|^{2H-2} \ du \ ds \right)^{\frac{1}{2}}
\]

\( \mathcal{M}(\mathcal{K}) \)  
The space of Gaussian martingales of the form \( M_t = \int_0^t a(s)dW_s, \) where \( a \in \mathcal{K} \subset L_2([0,T]) \)

\( \mathfrak{M}_f \)  
The set of minimizing functions for the functional \( f \) on \( L_2([0,1]) \)
of the form \( f(x) = \sup_{t \in [0,1]} \left( \int_0^t (z(t,s) - x(s))^2 \ ds \right)^{1/2} \)

\( \mathcal{N}(0,1) \)  
The standard normal distribution

\( \mathbb{N} \)  
The set of natural numbers, i.e. the positive integers

\( \mathbb{R} \)  
The set of real numbers

\( \mathbb{R}^+ = [0, \infty) \)

\( W_1^\beta[0,T] \)  
The space of measurable functions \( f : [0, T] \rightarrow \mathbb{R} \) such that

\[
\|f\|_{1,\beta} = \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \ du \right) < \infty,
\]
\[ W^\beta_2[0, T] \] The space of measurable functions \( f : [0, T] \to \mathbb{R} \) such that

\[
\| f \|_{2, \beta} := \int_0^T \frac{|f(s)|}{s^\beta} \, ds + \int_0^T \int_0^s \frac{|f(s) - f(u)|}{(s - u)^{\beta + 1}} \, du \, ds < \infty.
\]

\( w[J] = (w_i, i \in J) \), The vector made of the elements of vector \( w \) with indices within \( J \).

\( X[\cdot, J] \) The submatrix of the matrix \( X \) constructed of the columns of the matrix \( X \) with indices within the set \( J \);

\( x_+ = \max\{x, 0\} \)

\( z(t, s) \) The Molchan kernel

\( \Gamma(\alpha) \) The gamma function

\[
\rho^2_H(M) = \sup_{t \in [0, T]} E(B^H_t - M_t)^2
\]

\[
\rho^2(B^H, M(K)) = \inf_{a \in K} \sup_{t \in [0, T]} E(B^H_t - M_t)^2,
\]

where \( M_t = \int_0^t a(s) \, dW_s, \ K \subset L_2([0, T]) \)

\( (\Omega, \mathcal{F}, P) \) Complete probability space

\( \mathbf{1}_A \) Indicator function of a set \( A \)

\( = \) Equality in distribution (equality of all finite-dimensional distributions)

\( \xrightarrow{P} \) Convergence in probability
Fractional Brownian motion (fBm) $B^H = \{B^H_t, t \geq 0\}$ with Hurst index $H \in (0, 1)$ is a very interesting stochastic object that has attracted increased attention due to its peculiar properties. On the one hand, this is a Gaussian random process with a fairly simple covariance function that provides the Hölder property of trajectories up to the order $H$. On the other hand, it is a generalization of the Wiener process, which corresponds to the value of the Hurst index $H = 1/2$. Finally, it is neither a process with independent increments, nor a Markov process, nor a semimartingale unless $H = 1/2$, and therefore it can be used to model quite complex real processes that demonstrate the phenomenon of memory, both long and short. Long memory corresponds to $H > 1/2$, while short memory is inherent in $H < 1/2$. The combination of these properties is useful in modeling the processes occurring in devices that provide cellular and other types of communication, in physical and biological systems and in finance and insurance. Thus, the fBm itself deserves special attention. We will not discuss all the aspects of fBm here, and recommend the books [BIA 08, KUB 17, MIS 08, MIS 17, MIS 18, NOU 12, NUA 03, SAM 06] for more detail concerning various fractional processes.

Now note that the absence of semimartingale and Markov properties always causes the study of the possibility of the approximation of fBm by simpler processes, in a suitable metric. Without claiming a comprehensive review of the available results, we list the following studies: approximation of fBm by the continuous processes of bounded variation was studied in [AND 06, RAL 11b], approximating wavelets were considered in [AYA 03], weak convergence to fBm in the schemes of series of various sequences of processes was discussed in [GOR 78, NIE 04, TAQ 75] and some other studies, and summarized in [MIS 08]. The paper [MUR 11] contains a presentation of fBm in terms of an infinite-dimensional Ornstein–Uhlenbeck
process. The approximation of fBm by semimartingales is proposed in [DUN 11]. The article [RAL 11a] investigates smooth approximations for the so-called multifractional Brownian motion, a generalization of fBm to the case of time-varying Hurst index. Approximation of fBm using the Karhunen theorem and using various decompositions into series over functional bases is also investigated in great detail.

There is also such a question, which, in fact, served as the main incentive for writing this book: is it possible to approximate an fBm by martingales, in a reasonably chosen metric? If not, is it possible to find a projection of fBm on the class of martingales and the distance between fBm and this projection? Such a seemingly simple and easily formulated question actually led to, in our opinion, quite unexpected, non-standard and interesting results that we decided to offer them to the attention of the reader. Metric, which was proposed, has the following form:

$$\rho_H(M) := \sup_{t \in [0,T]} \left( \mathbb{E}(B^H_t - M_t)^2 \right)^{1/2},$$

where $M = \{M_t, t \in [0,T]\}$ is a martingale adapted to the filtration generated by $B^H$. So, we consider the distance in the space $L_\infty([0,T];L_2(\Omega))$. The first problem, considered in this book, is the minimization of $\rho_H(M)$ over the class of adapted martingales. Chapter 1 is fully devoted to this problem. We perform the following procedures step by step: introducing the so-called Molchan representation of fBm via Volterra kernel and the underlying Wiener process; proving that minimum is achieved within the class of martingales of the form $M_t = \int_0^t a(s) dW_s$, where $W$ is the underlying Wiener process and $a$ is a non-random function from $L_2([0,T])$. As a result, the minimization problem becomes analytical. Since it is essentially minimax problem, we used a convex analysis to establish the existence and uniqueness of minimizing function $a$. The existence follows from the convexity of the distance. However, the proof of the uniqueness essentially relies on self-similarity of fBm. If some other Gaussian process is considered instead of fBm, the minimum of the distance may be attained for multiple functions $a$. Then, we propose an original probabilistic representation of the minimizing function $a$ and establish several properties of this function. However, its analytical representation is unknown; therefore, the problem is to find its values numerically. In this connection, we considered a discrete-time counterpart of the minimization problem and reduced it, via iterative minimization using alternating minimization method, to the calculation of the Chebyshev center. It allows us to draw the plots of the minimizing function, as well as the plot of square distance between fBm and the space of adapted Gaussian martingales as a function of the Hurst index.
So, since the problem of finding a minimizing function in the whole class $L_2([0,T])$ turned out to be one that requires a numerical solution, and it is necessary to use fairly advanced methods, we then tried to minimize the distance of an fBm to the subclasses of martingales corresponding to simpler functions, in order to obtain an analytical solution, or numerical, but with simpler methods, without using tools of convex analysis. Since the Volterra kernel in the Molchan representation of fBm consists of power functions, it is natural to consider various subclasses of $L_2([0,T])$ consisting of power functions and their combinations. Even in this case, the problem of minimization is not easy and allows an explicit solution only in some cases, many of which are discussed in detail in Chapter 2. Somewhat unexpected, however, for some reason, natural, is the fact that the normalizing constant in the Volterra kernel, which usually does not play any role and is even often omitted, comes to the fore in calculations and, so to speak, directs the result. Moreover, in the course of calculations, interesting new relations were obtained for gamma functions and their combinations, and even a new upper bound for the cardinal sine function $\frac{\sin x}{x}$ was produced.

Chapter 3 is devoted to the approximations of fBm by various processes of comparatively simple structure. In particular, we represent fBm as a uniformly convergent series of Lebesgue integrals, describe the semimartingale approximation of fBm and propose a construction of absolutely continuous processes that converge to fBm in certain Besov-type spaces. Special attention is given to the approximation of pathwise stochastic integrals with respect to fBm. In the last section of this chapter, we study smooth approximations of multifractional Brownian motion.

Appendix 1 contains the necessary auxiliary facts from mathematical, functional and stochastic analyses, especially from the theory of gamma functions, elements of convex analysis, the Garsia–Rodemich–Rumsey inequality, basics of martingales and semimartingales and introduction to stochastic integration with respect to an fBm. Appendix 2 explains how to evaluate the Chebyshev center, together with pseudocode. In Appendix 3, we describe several techniques of fBm simulation. In particular, we consider in detail the Cholesky decomposition of the covariance matrix, the Hosking method (also known as the Durbin–Levinson algorithm) and the very efficient method of exact simulation via circulant embedding and fast Fourier transform. A more detailed description of the book’s content by section is at the beginning of each chapter.

The results presented in this book are based on the authors’ papers [BAN 08, BAN 11, BAN 15, DOR 13, MIS 09, RAL 10, RAL 11a, RAL 11b, RAL 12, SHK 14] as well as on the results from [DAV 87, DIE 02, DUN 11, HOS 84, SHE 15, WOO 94].
It is assumed that the reader is familiar with the basic concepts of mathematical analysis and the theory of random processes, but we tried to make the book self-contained, and therefore most of the necessary information is included in the text. This book will be of interest to a wide audience of readers; it is comprehensible to graduate students and even senior students, useful to specialists in both stochastics and convex analysis, and to everyone interested in fractional processes and their applications.

We are grateful to everyone who contributed to the creation of this book, especially to Georgiy Shevchenko, who is the author of the results concerning the probabilistic representation of the minimizing function.

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Consider the fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. Its definition and properties will be considered in more detail in section 1.1; however, let us mention immediately that fBm is a Gaussian process and anyhow not a martingale or even a semimartingale for $H \neq \frac{1}{2}$. Hence, a natural question arises: what is the distance between fBm and the space of Gaussian martingales in an appropriate metric and how do we determine the projection of fBm on the space of Gaussian martingales? Why is it not reasonable to consider non-Gaussian martingales? In this chapter, we will answer this and other related questions. The chapter is organized as follows. In section 1.1, we give the main properties of fBm, including its integral representations. In section 1.2, we formulate the minimizing problem simplifying it at the same time. In section 1.3, we strictly propose a positive lower bound for the distance between fBm and the space of Gaussian martingales. Sections 1.4 and 1.5 are devoted to the general problem of minimization of the functional $f$ on $L_2([0,1])$ that has the following form:

$$f(x) = \sup_{t \in [0,1]} \left( \int_0^t \left( z(t,s) - x(s) \right)^2 ds \right)^{1/2}$$

[1.1]

with arbitrary kernel $z(t,s)$ satisfying condition (A) for any $t \in [0,1]$ the kernel $z(t, \cdot) \in L_2([0,t])$ and

$$\sup_{t \in [0,1]} \int_0^t z(t,s)^2 \, ds < \infty.$$  

[1.2]
We shall call the functional \( f \) the *principal functional*. It is proved in section 1.4 that the principal functional \( f \) is convex, continuous and unbounded on infinity, consequently the minimum is reached. Section 1.5 gives an example of the kernel \( z(t, s) \) where a minimizing function for the principal functional is not unique (moreover, being convex, the set of minimizing functions is infinite). Sections 1.6–1.8 are devoted to the problem of minimization of principal functional \( f \) with the kernel \( z \) corresponding to fBm, i.e. with the kernel \( z \) from [1.7]. It is proved in section 1.6 that in this case, the minimizing function for the principal functional is unique. In section 1.7 it is proved that the minimizing function has a special form, namely a probabilistic representation, and many properties of the minimizing function have been established. Since we have no explicit analytical representation of the minimizing function, in section 1.8 we provide the discrete-time counterpart of the minimization problem and give the results explaining how to calculate the minimizing function numerically via evaluation of the Chebyshev center, illustrating the numerics with a couple of plots.

1.1. fBm and its integral representations

In this section, we define fBm and collect some of its main properties. We refer to the books [BIA 08, MIS 08, MIS 18, NOU 12] for the detailed presentation of this topic.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the standard assumptions.

**Definition 1.1.**– An fBm with associated Hurst index \( H \in (0, 1) \) is a Gaussian process \( B^H = \{B^H_t, \mathcal{F}_t, t \geq 0\} \), such that

1) \( E[B^H_t] = 0, \quad t \geq 0, \)

2) \( E[B^H_t B^H_s] = \frac{1}{2} \left( (t^{2H} + s^{2H}) - |t - s|^{2H} \right), \quad s, t \geq 0. \)

The following statements can be derived directly from the above definition.

1) If \( H = \frac{1}{2} \), then an fBm is a standard Wiener process.

2) An fBm is self-similar with the self-similarity parameter \( H \), i.e. \( \{B^H_t\} \overset{d}{=} \{c^H B^H_t\} \) for any \( c > 0 \). Here, \( \overset{d}{=} \) means that all finite-dimensional distributions of both processes coincide.

3) An fBm has stationary increments that is implied by the form of its incremental covariance:

\[
E(B^H_t - B^H_s)^2 = (t - s)^{2H}.
\]  

[1.3]
4) The increments of an fBm are independent only in the case $H = 1/2$. They are negatively correlated for $H \in (0, 1/2)$ and positively correlated for $H \in (1/2, 1)$.

Due to the Kolmogorov continuity theorem, property [1.3] implies that an fBm has a continuous modification. Moreover, this modification is $\gamma$-Hölder continuous on each finite interval for any $\gamma \in (0, H)$.

It is also well-known that an fBm is not a process of bounded variation. If $H \neq 1/2$, then it is neither a semimartingale nor a Markov process.

An fBm can be represented as an integral of a deterministic kernel with respect to the standard Wiener process in several ways.

We start with the **Molchan representation** (or Volterra-type representation) of fBm $B^H_t = \{B^H_t, \mathcal{F}_t, t \geq 0\}$ via the Wiener process on a finite interval (see, for example, [NOR 99b, NUA 03]). It states that a Wiener process $W = \{W_t, \mathcal{F}_t, t \geq 0\}$ exists, such that for any $t \geq 0$

$$B^H_t = \int_0^t z(t, s) \, dW_s, \tag{1.4}$$

where the Molchan kernel is defined by

$$z(t, s) = c_H \left( t^{H-1/2} s^{1/2-H} (t-s)^{H-1/2} 
- (H - \frac{1}{2}) s^{1/2-H} \int_s^t u^{H-3/2} (u-s)^{H-1/2} \, du \right) 1_{0 < s < t}, \tag{1.5}$$

with

$$c_H = \left( \frac{2H \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{1/2}. \tag{1.6}$$

In the case $H \in (\frac{1}{2}, 1)$, the kernel $z(t, s)$ can be simplified to

$$z(t, s) = c_H \left( H - \frac{1}{2} \right) s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} \, du 1_{0 < s < t}. \tag{1.7}$$

The **Mandelbrot–Van Ness representation**, or moving-average representation, was obtained in [MAN 68]. It states that an fBm $B^H_t$ can be represented as

$$B^H_t = \int_{-\infty}^t z_1(t, s) \, d\widetilde{W}_s, \tag{1.8}$$
where $\tilde{W}$ is a two-sided Wiener process, and the Volterra kernel $z_1$ is defined by the formula

$$z_1(t, s) = c_H \left( (t - s)_+^{H-1/2} - (-s)_+^{H-1/2} \right),$$

[1.9]

and $x_+ = \max\{x, 0\}$. This representation defines fBm on the whole axis, but in what follows we shall consider only the processes that are defined on $\mathbb{R}^+$. The next result demonstrates that the finite-dimensional distributions of fBm are of non-degenerate form.

**Theorem 1.1.** Let $0 < H < 1$, $B^H = \{B_t^H, t \geq 0\}$ be an fBm with associated Hurst index $H$. Then, the finite-dimensional distributions of $B^H$ have a non-singular covariance matrix, i.e., for any set of points $t_1, \ldots, t_n$, $0 < t_1 < \ldots < t_n$, the covariance matrix $Evv^\top$ of the slice-vector $v = (B_{t_1}^H, \ldots, B_{t_n}^H)$ is non-singular.

**Proof.** Recall that fBm $B^H$ admits the Mandelbrot–Van Ness representation [1.8]. Now, let $0 < t_1 < \ldots < t_n$. We carry out the proof by contradiction. Thus, assume that the vector $v = (B_{t_1}^H, \ldots, B_{t_n}^H)$, which has multivariate Gaussian distribution with zero mean, has a singular covariance matrix. Then,

$$E((vv^\top) \cdot (\alpha_1, \ldots, \alpha_n)^\top) = 0$$

for some non-zero vector $(\alpha_1, \ldots, \alpha_n)^\top$. In turn, it follows that

$$E((v^\top) \cdot (\alpha_1, \ldots, \alpha_n)^\top)^2 = 0,$$

or, that is the same:

$$E \left( \sum_{k=1}^n \alpha_k B_{t_k}^H \right)^2 = 0.$$  

[1.10]

Without loss of generality, we can assume that $\alpha_k \neq 0$ for all $k = 1, \ldots, n$. Denote $t_0 = 0$. With Mandelbrot–Van Ness representation [1.8], we obtain

$$\sum_{k=1}^n \alpha_k B_{t_k}^H = c_H \int_{-\infty}^{t_n} a(s) \, dW_s,$$

where:

$$a(s) = \sum_{k=1}^n \alpha_k (t_k - s)^{H-1/2} - \sum_{k=1}^n \alpha_k (-s)^{H-1/2}, \quad \text{for} \quad s < 0;$$
\[ a(s) = \sum_{k=m}^{n} \alpha_k (t_k - s)^{H-1/2}, \quad \text{for} \quad t_{m-1} < s < t_m, \quad m = 1, \ldots, n; \]
\[ a(s) = 0, \quad \text{for} \quad s > t_n. \]

Observe that \( a = a(s) \) is not an almost-everywhere (a.e.) zero function, particularly \( a(s) \neq 0 \) for all \( s \in (t_{n-1}, t_n) \). Hence, \( \int_{-\infty}^{t_n} a(s)^2 \, ds > 0 \), and
\[
E \left( \sum_{k=1}^{n} \alpha_k B_{t_k}^H \right)^2 = c_H^2 \int_{-\infty}^{t_n} a(s)^2 \, ds > 0.
\]

We got the contradiction with [1.10], whence the proof follows. \( \square \)

1.2. Formulation of the main problem

Let \( B^H = \{ B_t^H, \mathcal{F}_t, t \in [0, T] \} \) be an fBm with Hurst index \( H \in (0, 1) \), restricted to the interval \([0, T]\). Recall that an fBm is neither a semimartingale nor a Markov process unless \( H = 1/2 \). Therefore, a simple and natural question is: how far is Brownian motion from being a martingale? That is, in a sense, we look for the distance between fBm and the space of martingales and for the projection of fBm on the space of (square integrable) martingales. Thus, initially, the problem is formulated in such a way: we are looking for a square integrable \( \mathcal{F} \)-martingale \( M \) that minimizes the value
\[
\rho_H^2(M) := \sup_{t \in [0, T]} E \left( B_t^H - M_t \right)^2,
\]
and try to calculate or estimate the value itself. To proceed with the solution of this problem, we can use the Molchan representation [1.4] of the fBm \( B^H \) via the standard \( \mathcal{F} \)-Brownian motion \( W = \{ W_t, \mathcal{F}_t, t \in [0, T] \} \). We observe first that \( B^H \) and \( W \) generate the same filtration, and so according to the standard martingale representation theorem (see, for example, [DAV 05]), any square integrable \( \mathcal{F} \)-martingale \( M \) admits a representation
\[
M_t = \int_0^t a(s) \, dW_s, \quad [1.11]
\]
where \( a \) is an \( \mathcal{F} \)-adapted square integrable process.

Hence, we can write
\[
E \left( B_t^H - M_t \right)^2 = E \left( \int_0^t \left( z(t, s) - a(s) \right) \, dW_s \right)^2
\]
\[ = \int_0^t \mathbb{E}(z(t, s) - a(s))^2 \, ds \]
\[ = \int_0^t (z(t, s) - \mathbb{E}a(s))^2 \, ds + \int_0^t \text{Var} \, a(s) \, ds. \]

Consequently, it is enough to minimize \( \rho_H(M) \) over Gaussian martingales, i.e. those having representation [1.11] with a non-random \( a \).

Hence, the main problem reduces to the following one: take the functional \( f \) from [1.1] and find

\[ \rho^2(B^H, \mathcal{M}(L_2([0, T]))) := \inf_{x \in L_2([0, T])} \sup_{t \in [0, T]} \int_0^t (z(t, s) - x(s))^2 \, ds \]
\[ = \inf_{x \in L_2([0, T])} f(x), \tag{1.12} \]

and a minimizing element \( a \in L_2([0, T]) \) if the infimum is reached. Note that the expression being minimized involves neither an fBm nor a Wiener process; thus, the problem becomes purely analytic.

Note that it is natural to consider the integral in [1.11] with respect to the Wiener process \( W \) from the Molchan representation of \( B^H \), which can be called the underlying Wiener process. Indeed, the distance \( \mathbb{E}(B^H_t - M_t)^2 \) increases if \( M_t \) is of the form \( M_t = \int_0^t a(s) \, d\tilde{W}_s \), where \( \tilde{W} \) is a Wiener process with a component independent of \( W \). This fact is established in the following lemma.

**Lemma 1.1.** Among all Wiener processes \( \left\{ \tilde{W}_t, \mathcal{F}_t, t \in [0, T] \right\} \), the minimum in the expression \( \mathbb{E}\left(B^H_t - \int_0^t a(s) \, d\tilde{W}_s\right)^2 \) is reached for

\[ \tilde{W}_t = \int_0^t \text{sgn} \, a(s) \, dW_s, \]

where \( W \) is a Wiener process from the Molchan representation [1.4] of \( B^H \).

**Proof.** Let \( \left\{ \tilde{W}_t, \mathcal{F}_t, t \in [0, T] \right\} \) be a Wiener process correlated with \( W \) so that

\[ \mathbb{E}\tilde{W}_t W_t = \rho(t). \]

Then,

\[ \rho(t) - \rho(s) = \mathbb{E}\left(\tilde{W}_t - \tilde{W}_s\right) (W_t - W_s) + \mathbb{E}W_s (W_t - W_s) + \mathbb{E}\left(\tilde{W}_t - \tilde{W}_s\right) W_s. \]
By conditioning with respect to $\mathcal{F}_s$, we can easily show that for all $t > s$, $\mathbb{E}\tilde{W}_s(W_t - W_s) = \mathbb{E}\left(\tilde{W}_t - \tilde{W}_s\right)W_s = 0$. Hence, by the Cauchy–Schwarz inequality,

$$|\rho(t) - \rho(s)| \leq \sqrt{\mathbb{E}\left(\tilde{W}_t - \tilde{W}_s\right)^2\mathbb{E}(W_t - W_s)^2} = |t - s|.$$ 

Therefore, $\rho(t) = \int_0^t \theta(s)\, ds$ with $|\theta(s)| \leq 1$, and

$$\tilde{W}_t = \int_0^t \theta(s)\, dW_s + \int_0^t \sqrt{1 - \theta^2(s)}\, dZ_s,$$

where $Z = \{Z_t, \mathcal{F}_t, t \in [0, T]\}$ is a Wiener process independent of $W$. Then,

$$\mathbb{E}\left(B_t^H - \int_0^t a(s)\, d\tilde{W}_s\right)^2 = \mathbb{E}\left(\int_0^t z(t, s)\, dW_s - \int_0^t a(s)\theta(s)\, dW_s - \int_0^t a(s)\sqrt{1 - \theta^2(s)}\, dZ_s\right)^2 = \mathbb{E}\left(\int_0^t z(t, s)\, dW_s - \int_0^t a(s)\theta(s)\, dW_s\right)^2 + \int_0^t a^2(s)\left(1 - \theta^2(s)\right)\, ds = \int_0^t (z(t, s) - a(s)\theta(s))^2\, ds + \int_0^t a^2(s)\left(1 - \theta^2(s)\right)\, ds = t^{2H} - 2\int_0^t a(s)\theta(s)z(t, s)\, ds + \int_0^t a^2(s)\, ds. \quad [1.13]$$

Since $z(t, s) > 0$, we see that the minimum in [1.13] is reached at the function

$$a(s)\theta(s) = |a(s)|,$$

i.e. $\theta(s) = \text{sgn} a(s)$. □

**Corollary 1.1.** – Since the proof of Lemma 1.1 is valid for any non-negative kernel $z$, satisfying condition [1.2], from now on, considering the non-negative kernel $z$, we can restrict ourselves to Gaussian martingales of the form $M_t = \int_0^t a(s)\, dW_s$, where $W$ is the underlying Wiener process, and function $a$ is non-negative.
1.3. The lower bound for the distance between fBm and Gaussian martingales

Denote $\mathcal{M}(\mathcal{K})$ the space of the Gaussian martingales of the form $M_t = \int_0^t a(s) dW_s$, where $a \in \mathcal{K} \subset L_2([0,T])$. In the following theorem, using stochastic considerations, we establish a non-zero lower bound for the distance between fBm with the Molchan kernel and the space of all Gaussian martingales, i.e. the space $\mathcal{M}(L_2([0,T]))$. All processes are considered on the fixed interval $[0,T]$.

**Theorem 1.2.** The value

$$\rho_T := \rho^2(B^H, \mathcal{M}(L_2([0,T]))) = \inf_{a \in L_2([0,T])} \sup_{0 \leq t \leq T} E \left( B^H_t - \int_0^t a(s) dW_s \right)^2$$

admits the following lower bound:

$$\rho_T \geq \max_{0 \leq t \leq 1} \frac{(1-t^{2H})^2 - (1-t)^{2H}}{16t^{2H}} \cdot T^{2H} > 0. \quad [1.14]$$

**Proof.** Note that our kernel $z(t,s)$ is homogeneous in the following sense:

$$z(t,s) = T^{H-1/2} z(t/T, s/T).$$

Therefore,

$$\sup_{0 \leq t \leq T} E \left( B^H_t - \int_0^t a(s) dW_s \right)^2 = \sup_{0 \leq t \leq T} E \int_0^t (z(t,s) - a(s))^2 ds$$

$$= T^{2H-1} \sup_{0 \leq t \leq T} E \int_0^{t/T \cdot T} (z(t/T,s/T) - a(s/T \cdot T)^{1/2-H})^2 ds$$

$$= T^{2H} \sup_{0 \leq u \leq 1} E \int_0^u (z(u,v) - a(v \cdot T)^{1/2-H})^2 dv,$$

and consequently $\rho_T$ can be rewritten via $\rho_1$, namely $\rho_T = T^{2H} \rho_1$. This implies that we can restrict the consideration to the case of $\rho_T$ with $T = 1$. Now we construct a lower bound for

$$\max_{0 \leq t \leq 1} E \left( B^H_t - \int_0^t a(s) dW_s \right)^2 = \max_{0 \leq t \leq 1} \int_0^t (z(t,s) - a(s))^2 ds.$$
Let $0 < t_1 \leq 1$. Consider the random variable $\int_0^1 a(s) dW_s =: B$. Then,

\[
\max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2 \geq \max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \mathbb{E}[B | \mathcal{F}_t] \right)^2
\]

Now we can use variance partitioning. Obviously, for every square integrable random variable $\eta$, we have

\[
\mathbb{E} \left[ (\eta - \mathbb{E}[\eta | B_{t_1}])^2 | B_{t_1} \right] = \mathbb{E}[\eta^2 | B_{t_1}] - (\mathbb{E}[\eta | B_{t_1}])^2.
\]

Hence,

\[
\mathbb{E}(\eta - \mathbb{E}[\eta | B_{t_1}])^2 = \mathbb{E}\eta^2 - \mathbb{E}(\mathbb{E}[\eta | B_{t_1}])^2.
\]

We apply the inequality $\mathbb{E}\eta^2 \geq \mathbb{E}(\mathbb{E}[\eta | B_{t_1}])^2$ for $\eta = B_{t_1}^H - \mathbb{E}[B | \mathcal{F}_{t_1}]$ and for $\eta = B_{1}^H - B$, and obtain

\[
\max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2 \geq \frac{1}{2} \left( \mathbb{E} \left( B_{t_1}^H - \mathbb{E} \left[ B \mid B_{t_1}^H \right] \right)^2 + \mathbb{E} \left( \mathbb{E} \left[ B_{1}^H \mid B_{t_1}^H \right] - \mathbb{E} \left[ B \mid B_{t_1}^H \right] \right)^2 \right).
\]

Note that for all real numbers $P$, $Q$ and $r$, the inequality

\[
\frac{(P - r)^2}{2} + \frac{(Q - r)^2}{2} \geq \frac{(P - Q)^2}{4}
\]

holds true because $2(P - r)^2 + 2(Q - r)^2 - (P - Q)^2 = (P + Q - 2r)^2 \geq 0$. Therefore,

\[
\max_{0 \leq t \leq 1} \mathbb{E} \left( B_t^H - \int_0^t a(s) dW_s \right)^2 \geq \frac{1}{4} \mathbb{E} \left( B_{t_1}^H - \mathbb{E} \left[ B_{1}^H \mid B_{t_1}^H \right] \right)^2 = \frac{1}{4} \mathbb{E} \left( B_{t_1}^H - \frac{\mathbb{E}(B_{1}^H B_{t_1}^H)}{\mathbb{E}(B_{t_1}^H)^2} B_{t_1}^H \right)^2.
\]
\[
\begin{align*}
E \left[ B_{t_1}^{H} \left( 1 - \frac{1 + t_1^{2H} - (1 - t_1)^{2H}}{2t_1^{2H}} \right) \right]^2 &= \frac{1}{4} \left( 1 - \frac{1 + t_1^{2H} - (1 - t_1)^{2H}}{2t_1^{2H}} \right)^2 \\
&= \frac{1}{4} t_1^{2H} \left( 1 - \frac{1 + t_1^{2H} - (1 - t_1)^{2H}}{2t_1^{2H}} \right)^2 = \frac{1 - t_1^{2H} - (1 - t_1)^{2H}}{16t_1^{2H}}.
\end{align*}
\]

**Remark 1.1.**— By substituting \( t = \frac{1}{2} \) into maximized expression in [1.14], we obtain a loose lower bound
\[
\rho_T \geq \frac{\left( 2^{2H} - 2 \right)^2}{16 \cdot 2^{2H}} \cdot T^{2H}.
\]

### 1.4. The existence of minimizing function for the principal functional

Recall that in the analytic form, our goal is to find

\[
\inf_{x \in L_2([0,T])} \sup_{t \in [0,T]} \int_0^t (z(t,s) - x(s))^2 \, ds = \inf_{x \in L_2([0,T])} f(x),
\]

where the functional \( f = f(x) \) is defined via [1.1], and a minimizing element \( x \in L_2([0,T]) \) if the infimum is reached. In the original formulation, \( z \) is the kernel related to an fBm. However, the solution of this problem is based on the general properties of functionals in a Hilbert space, in particular, functionals defined by kernels satisfying the assumption (A) with inequality [1.2]. Therefore, in this section, we consider arbitrary kernel \( z \) satisfying assumption (A), which implies that the functional \( f = f(x) \) is well defined for any \( x \in L_2([0,1]) \). From this point on, with a view to simplifying the computations, let us consider in this chapter only the case \( T = 1 \).

**Lemma 1.2.**— *For any \( x, y \in L_2([0,1]) \),

\[
|f(x) - f(y)| \leq \|x - y\|_{L_2([0,1])}.
\]

**Proof.**— Evidently, for any \( x, y \in L_2([0,1]) \) and \( 0 \leq t \leq 1 \),

\[
\left( \int_0^t (z(t,s) - x(s))^2 \, ds \right)^{1/2} \leq \left( \int_0^t (x(s) - y(s))^2 \, ds \right)^{1/2} + \left( \int_0^t (z(t,s) - y(s))^2 \, ds \right)^{1/2}.
\]
Therefore,
\[
\begin{aligned}
\sup_{t \in [0,1]} \left( \int_0^t (z(t,s) - x(s))^2 \, ds \right)^{1/2} \\
\leq \sup_{t \in [0,1]} \left( \int_0^t (x(s) - y(s))^2 \, ds \right)^{1/2} + \sup_{t \in [0,1]} \left( \int_0^t (z(t,s) - y(s))^2 \, ds \right)^{1/2},
\end{aligned}
\]
which is clearly equivalent to the inequality
\[
f(x) \leq \|x - y\|_{L_2([0,1])} + f(y).
\]

Swapping \(x\) and \(y\), we establish [1.16] and thus obtain the proof.

**Corollary 1.2.** The functional \(f\) is continuous on \(L_2([0,1])\).

**Lemma 1.3.** The following inequalities hold for any function \(x \in L_2([0,1])\):
\[
|x|_{L_2([0,1])} - \|z(1,\cdot)\|_{L_2([0,1])} \leq f(x) \leq \|x\|_{L_2([0,1])} + f(0). \tag{1.17}
\]

**Proof.** The left-hand side of [1.17] immediately follows from the inequalities
\[
f(x) \geq \left( \int_0^1 (z(1,s) - x(s))^2 \, ds \right)^{1/2} = \|z(1,\cdot) - x\|_{L_2([0,1])}
\]
\[
\geq |x|_{L_2([0,1])} - \|z(1,\cdot)\|_{L_2([0,1])},
\]
and the right-hand side of [1.17] follows from [1.16].

**Lemma 1.4.** The functional \(f\) is convex on \(L_2([0,1])\).

**Proof.** We have to prove that for any \(x, y \in L_2([0,1])\) and any \(\alpha \in [0,1]\),
\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y). \tag{1.18}
\]
Applying the triangle inequality, we note that for any \(t \in [0,1]\)
\[
\left( \int_0^t [\alpha x(s) + (1 - \alpha)y(s) - z(t,s)]^2 \, ds \right)^{1/2}
\]
\[
\leq \left( \int_0^t [\alpha z(t,s) - x(s)]^2 \, ds \right)^{1/2} + \left( \int_0^t [(1 - \alpha)z(t,s) - y(s)]^2 \, ds \right)^{1/2},
\]
whence
\[
\sup_{t \in [0,1]} \left( \int_0^t \left[ \alpha x(s) + (1 - \alpha) y(s) - z(t, s) \right]^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \alpha \sup_{t \in [0,1]} \left( \int_0^t \left( z(t, s) - x(s) \right)^2 ds \right)^{\frac{1}{2}}
\]
\[
+ (1 - \alpha) \sup_{t \in [0,1]} \left( \int_0^t \left( z(t, s) - y(s) \right)^2 ds \right)^{\frac{1}{2}},
\]
and inequality [1.18] follows. \(\square\)

**Theorem 1.3.**— The functional \(f\) reaches its minimal value on \(L^2([0,1])\).

**Proof.**— By Corollary 1.2 and Lemma 1.4 the functional \(f\) is continuous and convex. By Lemma 1.3, \(f(x)\) tends to \(+\infty\) as \(\|x\| \to \infty\). Hence, it follows from Proposition A1.2 that \(f\) reaches its minimal value. \(\square\)

### 1.5. An example of the principal functional with infinite set of minimizing functions

We continue to study arbitrary kernel \(z\) satisfying assumption \((A)\), which implies that the functional \(f\) is well defined for any \(x \in L^2([0,1])\). Note that the set \(\mathcal{M}_f\) of minimizing functions for the functional \(f\) is convex. In this section, we consider an example of kernel \(z\) for which \(\mathcal{M}_f\) contains more than one point and consequently is infinite. First, we establish the following lower bound for the functional \(f\), which is similar to the particular case, considered in Theorem 1.2.

**Lemma 1.5.**— 1) Let the kernel \(z\) of the functional \(f\) defined by [1.1] satisfy assumption \((A)\). Then for any \(a \in L^2([0,1])\) and \(0 \leq t_1 < t_2 \leq 1\), the following inequality holds
\[
\sup_{t \in [0,1]} \int_0^t \left( z(t, s) - a(s) \right)^2 ds \geq \frac{1}{4} \int_0^{t_1} \left( z(t_2, s) - z(t_1, s) \right)^2 ds. \tag{1.19}
\]
2) The equality in [1.19] implies that
\[
a(s) = \frac{1}{2} \left( z(t_1, s) + z(t_2, s) \right) \quad a.e. \text{ on } [0,t_1), \tag{1.20}
\]
and
\[
a(s) = z(t_2, s) \quad a.e. \text{ on } [t_1,t_2]. \tag{1.21}
\]