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Mittag-Leffler Functions, Related Topics and Applications

Second Edition

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To the memory of our colleagues and friends

Anatoly Kilbas (1948–2010) and
Rudolf Gorenflo (1930–2017)

Preface to the Second Edition

After the appearance of the first edition of our book “Mittag-Leffler Functions: Related Topics and Applications”, we have observed a growing interest in the subject. Many new research articles and books have appeared. This is mainly due to the central role of the Mittag-Leffler functions in Fractional Calculus and Fractional Modeling. With this interest in mind, we decided to prepare the second edition of our book on Mittag-Leffler functions, presenting new ideas and results related to the theory and applications of this family of functions.

New results have been added to practically all sections of the book. In Chap. 3 “The Classical Mittag-Leffler Function”, results on Mittag-Leffler summation as well as the notion of the Mittag-Leffler reproducing kernel Hilbert space are discussed. We present results applying the distribution of the zeros of the Mittag-Leffler function to the study of inverse problems for differential equations in Banach spaces (Chap. 4 “The Two-Parametric Mittag-Leffler Function”). Chapter 5 “Mittag-Leffler Functions With Three Parameters” discusses recent results on Le Roy type functions related to the Mittag-Leffler function but having a different nature. New applications related to all functions in this chapter have been added. Essentially enlarged is the next chapter (Chap. 6) concerning Mittag-Leffler functions depending on several parameters. Such functions have become important from both a theoretical and an applied point of view. We also discuss the properties of the Mittag-Leffler functions of several variables and with matrix argument. Numerical methods for these classes of functions are discussed too. We have completely rewritten the chapters dealing with applications (Chap. 8 “Applications to Fractional Order Equations”, Chap. 9 “Applications to Deterministic Models”, and Chap. 10 “Applications to Stochastic Models”, which are essentially enlarged versions of Chaps. 7–9 from the first edition). Thus, we briefly discuss in Chap. 8 the main ideas of fractional control theory and present some numerical methods applied to the study of fractional models, including those related to the calculation of the values of the Mittag-Leffler functions. We also added a new chapter (Chap. 7), which describes the main properties of the classical Wright function, closely related to the Mittag-Leffler function. Consequently, the structure of Appendix F “Higher Transcendental Functions” has been changed, since now in App. F. 2 we deal mainly with the generalized Wright function, focussing not only on the general function of this type (the so-called Fox–Wright function ${}_pW_q$) but also on the most applicable special cases ${}_1W_1$ and ${}_0W_2$. Essential changes were also made to Appendix E "Elements of Fractional Calculus" in order to outline the role of less popular fractional constructions and to show which specific properties of these constructions give potential further applications of Grünwald–Letnikov, Marchaud, Hadamard, Erdélyi–Kober, and Riesz fractional derivatives.

Our book project could not have been realized without the constant support of our colleagues and friends. We are grateful to Roberto Garrappa for preparing short reviews of his results and allowing us to include them in the book. Additional thanks are due to Alexander Apelblat, Roberto Garra, Andrea Giusti, George Karneadakis, Virginia Kiryakova, Yuri Luchko, Arak Mathai, Edmundo Capelas de Oliveira, Gianni Pagnini, Enrico Scalas, José Tenreiro Machado, and Vladimir Uchaikin. Our wives, Giovanna and Maryna, were so polite to allow us to spend so much time on the book. The 2nd edition was discussed in Bologna and Berlin by three of us, but as of October 20, 2017, Professor Rudolf Gorenflo is no longer with us. We took the liberty to dedicate this second edition to our missed colleagues and friends, Anatoly Kilbas and Rudolf Gorenflo, keeping them as co-authors because of their essential role in realizing this project.

Bologna, Italy | Francesco Mainardi
Minsk, Belarus | Sergei Rogosin
March 2020

Preface to the First Edition

The study of the Mittag-Leffler function and its various generalizations has become a very popular topic in Mathematics and its Applications. However, during the twentieth century, this function was practically unknown to the majority of scientists, since it was ignored in most common books on special functions. As a noteworthy exception the handbook “Higher Transcendental Functions”, vol. 3, by A. Erdelyi et al. deserves to be mentioned.

Now the Mittag-Leffler function is leaving its isolated role as *Cinderella* (using the term coined by F.G. Tricomi for the *incomplete gamma* function).

The recent growing interest in this function is mainly due to its close relation to the *Fractional Calculus* and especially to fractional problems which come from applications.

Our decision to write this book was motivated by the need to fill the gap in the literature concerning this function, to explain its role in modern pure and applied mathematics, and to give the reader an idea of how one can use such a function in the investigation of modern problems from different scientific disciplines.

This book is a fruit of collaboration between researchers in Berlin, Bologna and Minsk. It has highly profited from visits of SR to the Department of Physics at the University of Bologna and from several visits of RG to Bologna and FM to the Department of Mathematics and Computer Science at Berlin Free University under the European ERASMUS exchange. RG and SR appreciate the deep scientific atmosphere at the University of Bologna and the perfect conditions they met there for intensive research.

We are saddened that our esteemed and always enthusiastic co-author Anatoly A. Kilbas is no longer with us, having lost his life in a tragic accident on 28 June 2010 in the South of Russia. We will keep him, and our inspiring joint work with him, in living memory.

Berlin, Germany
Bologna, Italy
Minsk, Belarus
March 2014

Rudolf Gorenflo
Francesco Mainardi
Sergei Rogosin

Contents

Chapter 1
Introduction

This book is devoted to an extended description of the properties of the Mittag-Leffler function, its numerous generalizations and their applications in different areas of modern science.

The function $E_\alpha(z)$ is named after the great Swedish mathematician **Gösta Magnus Mittag-Leffler** (1846–1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C},\ \mathrm{Re}\,\alpha > 0, \tag{1.0.1}$$

and studied its properties in 1902–1905 in five subsequent notes [ML1, ML2, ML3, ML4, ML5-5] in connection with his summation method for divergent series.

This function provides a simple generalization of the exponential function because of the replacement of $k! = \Gamma(k+1)$ by $(\alpha k)! = \Gamma(\alpha k + 1)$ in the denominator of the power terms of the exponential series.

During the first half of the twentieth century the Mittag-Leffler function remained almost unknown to the majority of scientists. They unjustly ignored it in many treatises on special functions, including the most popular (Abramowitz and Stegun [AbrSte72] and its new version "NIST Handbook of Mathematical Functions" [NIST]). Furthermore, there appeared some relevant works where the authors arrived at series or integral representations of this function without recognizing it, e.g., (Gnedenko and Kovalenko [GneKov68]), (Balakrishnan [BalV85]) and (Sanz-Serna [San88]). A description of the most important properties of this function is present in the third volume [ErdBat-3] of the Handbook on Higher Transcendental Functions of the Bateman Project, (Erdelyi et al.). In it, the authors have included the Mittag-Leffler functions in their Chapter XVIII devoted to the so-called miscellaneous functions. The attribution of 'miscellaneous' to the Mittag-Leffler function is due to the fact that it was only later, in the sixties, that it was recognized to belong to

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a more general class of higher transcendental functions, known as Fox H-functions (see, e.g., [MatSax78, KilSai04, MaSaHa10]). In fact, this class was well-established only after the seminal paper by Fox [Fox61]. A more detailed account of the Mittag-Leffler function is given in the treatise on complex functions by Sansone and Gerretsen [SanGer60]. However, the most specialized treatise, where more details on the functions of Mittag-Leffler type are given, is surely the book by Dzherbashyan [Dzh66], in Russian. Unfortunately, no official English translation of this book is presently available. Nevertheless, Dzherbashyan has done a lot to popularize the Mittag-Leffler function from the point of view of its special role among entire functions of a complex variable, where this function can be considered as the simplest non-trivial generalization of the exponential function.

Successful applications of the Mittag-Leffler function and its generalizations, and their direct involvement in problems of physics, biology, chemistry, engineering and other applied sciences in recent decades has made them better known among scientists. A considerable literature is devoted to the investigation of the analyticity properties of these functions; in the references we quote several authors who, after Mittag-Leffler, have investigated such functions from a mathematical point of view. At last, the 2000 Mathematics Subject Classification has included these functions in item 33E12: "Mittag-Leffler functions and generalizations".

Starting from the classical paper of Hille and Tamarkin [HilTam30] in which the solution of Abel integral equation of the second kind

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} \mathrm{d}t = f(x), \ 0 < \alpha < 1, \quad 0 < x < 1, \tag{1.0.2}$$

is presented in terms of the Mittag-Leffler function, this function has become very important in the study of different types of integral equations. We should also mention the 1954 paper by Barret [Barr54], which was concerned with the general solution of the linear fractional differential equation with constant coefficients.

But the real importance of this function was recognized when its special role in fractional calculus was discovered (see, e.g., [SaKiMa93]). In recent times the attention of mathematicians and applied scientists towards the functions of Mittag-Leffler type has increased, overall because of their relation to the Fractional Calculus and its applications. Because the Fractional Calculus has attracted wide interest in different areas of applied sciences, we think that the Mittag-Leffler function is now beginning to leave behind its isolated life as Cinderella. We like to refer to the classical Mittag-Leffler function as the Queen Function of Fractional Calculus, and to consider all the related functions as her court.

A considerable literature is devoted to the investigation of the analytical properties of this function. In the references, in addition to purely mathematical investigations, we also mention several monographs, surveys and research articles dealing with different kinds of applications of the higher transcendental functions related to the Mittag-Leffler function. However, we have to point out once more that there exists no treatise specially devoted to the Mittag-Leffler function itself. In our opinion, it is now time for a book aimed at a wide audience. This book has to serve both as a textbook for beginners, describing the basic ideas and results in the area, and as a table-book for applied scientists in which they can find the most important facts for applications, and it should also be a good source for experts in Analysis and Applications, collecting deep results widely spread in the special literature. These ideas have been implemented into our plan for the present book. Because of the relevance of the Mittag-Leffler function to the theory and applications of Fractional Calculus we were invited to write a survey chapter for the first volume of the Handbook of Fractional Calculus with Applications [HAND1]. This chapter [GoMaRo19] presents in condensed form the main ideas of our book.

The book has the following structure. It can be formally considered as consisting of four main parts. The first part (INTRODUCTION AND HISTORY) consists of two chapters. The second part (THEORY) presents different aspects of the theory of the Mittag-Leffler function and its generalizations, in particular those arising in applied models. This part is divided into five chapters. The third part (APPLICATIONS) deals with different kinds of applications involving the Mittag-Leffler function and its generalizations. This part is divided into three chapters. Since the variety of models related to the Mittag-Leffler function is very large and rapidly growing, we mainly focus on how to use this function in different situations. We also separate theoretical applications dealing mainly with the solution of certain equations in terms of the Mittag-Leffler function from the more "practical" applications related to its use in modelling. Most of the auxiliary facts are collected in the fourth part consisting of six APPENDICES. The role of the appendices is multi-fold. First, we present those results which are helpful in reading the main text. Secondly, we discuss in part the machinery which can be omitted at the first reading of the corresponding chapter. Lastly, the appendices partly play the role of a handbook on some auxiliary subjects related to the Mittag-Leffler function. In this sense these appendices can be used to further develop the ideas contained in our book and in the references mentioned in it.

Each structural part of the book (either chapter or appendix) ends with a special section "Historical and Bibliographical Notes". We hope that these sections will help the readers to understand the features of the Mittag-Leffler function more deeply. We also hope that acquaintance with the book will give the readers new practical instruments for their research. In addition, since one of the aims of the book is to attract students, we present at the end of each chapter and each appendix a collection of exercises connected with different aspects of the theory and applications. Special attention is paid to the list of references which we have tried to make as complete as possible. Only seldom does the main text give references to the literature, the references are mainly deferred to the notes sections at the end of chapters and appendices. The bibliography contains a remarkably large number of references to articles and books not mentioned in the text, since they have attracted the author's attention over the last few decades and cover topics more or less related to this monograph. In the second edition we have significantly updated the bibliography. The interested reader will hopefully take advantage of this bibliography, enlarging and improving the scope of the monograph itself and developing new results.

Chapter 2 has in a sense a historical nature. We present here a few bibliographical notes about the creator of this book's subject, G.M. Mittag-Leffler. The contents of his pioneering works on the considered function is given here together with a brief description of the further development of the theory of the Mittag-Leffler function and its generalizations.

Chapter 3 is devoted to the classical Mittag-Leffler function (1.0.1). We collect here the main results on the function which were discovered during the century following Mittag-Leffler's definition. These are of an analytic nature, comprising rules of composition and asymptotic properties, and its character as an entire function of a complex variable. Special attention is paid to integral transforms related to the Mittag-Leffler function because of their importance in the solution of integral and differential equations. We point out its role in the Fractional Calculus and its place among the whole collection of higher transcendental functions.

In Chap. 4 we discuss questions similar to those of Chap. 3. This chapter deals with the simplest (and for applications most important) generalizations of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C},\ \mathrm{Re}\,\alpha > 0, \tag{1.0.3}$$

which was deeply investigated independently by Humbert and Agarwal in 1953 [Hum53, Aga53, HumAga53] and by Dzherbashyan in 1954 [Dzh54a, Dzh54b, Dzh54c] (but formally appeared first in the paper by Wiman [Wim05a]).

Chapter 4 presents the theory of two types of three-parametric Mittag-Leffler function. First of all it is the three-parametric Mittag-Leffler function (or Prabhakar function) introduced by Prabhakar [Pra71]

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k + \beta)} z^k, \alpha, \beta, \gamma \in \mathbb{C},\ \mathrm{Re}\,\alpha, \gamma > 0, \tag{1.0.4}$$

where $(\gamma)_k = \gamma(\gamma+1)\ldots(\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ is the Pochhammer symbol (see (A.17) in Appendix A). This function is now widely used for different applied problems. Another type of three-parametric Mittag-Leffler function is not as well-known as the Prabhakar function (1.0.4). It was introduced and studied by Kilbas and Saigo [KilSai95b] in connection with the solution of a new type of fractional differential equation. This function (the Kilbas–Saigo function) is defined as follows

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k \ (z \in \mathbb{C}), \tag{1.0.5}$$

where

$$c_0 = 1, \quad c_k = \prod_{i=1}^{k-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)} \ (k = 1, 2, \cdots), \ \alpha \in \mathbb{C}, \ \operatorname{Re}\alpha > 0. \tag{1.0.6}$$

Some basic results on this function are also included in Chap. 5. In the second edition we also include in this chapter another three-parametric generalization of the Mittag-Leffler function, namely, the Le Roy type function

$$F_{\alpha,\beta}^{(\gamma)} := \sum_{k=1}^{\infty} \frac{z^k}{(\Gamma(\alpha k + \beta))^{\gamma}}, \tag{1.0.7}$$

which is a function of a different nature than the other functions in this chapter. The Le Roy type function is a simple generalization of the Le Roy function [LeR00], which appeared as a competitor of the Mittag-Leffler function in the study of divergent series.

By introducing additional parameters one can discover new interesting properties of these functions (discussed in Chaps. 3–5) and extend their range of applicability. This is exactly the case with the generalizations described in this chapter. Together with some appendices, the above mentioned chapters constitute a short course on the Mittag-Leffler function and its generalizations. This course is self-contained and requires only a basic knowledge of Real and Complex Analysis.

Chapter 6 is rooted deeper mathematically. The reader can find here a number of modern generalizations. The ideas leading to them are described in detail. The main focus is on four-parametric Mittag-Leffler functions (Dzherbashyan [Dzh60]) and $2n$-parametric Mittag-Leffler functions (Al-Bassam and Luchko [Al-BLuc95] and Kiryakova [Kir99]). Experts in higher transcendental functions and their applications will find here many interesting results, obtained recently by various authors. These generalizations will all be labelled by the name Mittag-Leffler, in spite of the fact that some of them can be considered for many values of parameters as particular cases of the general class of Fox H-functions. These H-functions offer a powerful tool for formally solving many problems, however by inserting relevant parameters one often arrives at functions whose behavior is easier to handle. This is the case for the Mittag-Leffler functions, and so these functions are often more appropriate for applied scientists who prefer direct work to a detour through a wide field of generalities.

In the second edition we have decided to give a much wider presentation (new Chap. 7) of the classical Wright function

$$\phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}, \ \alpha > -1, \beta \in \mathbb{C}. \tag{1.0.8}$$

This function is closely related to the Mittag-Leffler function (especially to the four-parametric Mittag-Leffler function, see, e.g., [GoLuMa99, RogKor10]) and is of great importance for Fractional Calculus too. In spite of this similarity, some properties of the Wright function are not completely analogous to those of the Mittag-Leffler function. Thus, we found it important to discuss here the properties of the Wright function in detail.

The last three chapters deal with applications of the functions treated in the preceding chapters. We start (Chap. 8) with the "formal" (or mathematical) applications of Mittag-Leffler functions. The title of the chapter is "Applications to Fractional Order Equations". By fractional order equations we mean either integral equations with weak singularities or differential equations with ordinary or partial fractional derivatives. The collection of such equations involving Mittag-Leffler functions in their analysis or in their explicit solution is fairly big. Of course, we should note that a large number of fractional order equations arise in certain applied problems. We would like to separate the questions of mathematical analysis (solvability, asymptotics of solutions, their explicit presentation etc.) from the motivation and description of those models in which such equations arise. In Chap. 8 we focus on the development of a special "fractional" technique and give the reader an idea of how this technique can be applied in practice.

Further applications are presented in the two subsequent chapters devoted to mathematical modelling of special processes of interest in the applied sciences. Chapter 9 deals mainly with the role of Mittag-Leffler functions in discovering and analyzing deterministic models based on certain equations of fractional order. Special attention is paid to fractional relaxation and oscillation phenomena, to fractional diffusion and diffusive wave phenomena, to fractional models in dielectrics, models of particle motion in a viscous fluid, and to hereditary phenomena in visco-elasticity and hydrodynamics. These are models in physics, chemistry, biology etc., which by adopting a macroscopic viewpoint can be described without using probabilistic ideas and machinery.

In contrast, in Chap. 10 we describe the role of Mittag-Leffler functions in models involving randomness. We explain here the key role of probability distributions of Mittag-Leffler type which enter into a variety of stochastic processes, including fractional Poisson processes and the transition from continuous time random walk to fractional diffusion.

Our six appendices can be divided into two groups. First of all we present here some basic facts from certain areas of analysis. Such appendices are useful additions to the course of lectures which can be extracted from Chaps. 3–5. The second type of appendices constitute those which can help the reader to understand modern results in the areas in which the Mittag-Leffler function is essential and important. They serve to make the book self-contained.

The book is addressed to a wide audience. Special attention is paid to those topics which are accessible for students in Mathematics, Physics, Chemistry, Biology and Mathematical Economics. Also in our audience are experts in the theory of the Mittag-Leffler function and its applications. We hope that they will find the technical parts of the book and the historical and bibliographical remarks to be a source of new ideas. Lastly, we have to note that our main goal, which we always had in mind during the writing of the book, was to make it useful for people working in different areas of applications (even those far from pure mathematics).

Chapter 2
Historical Overview of the Mittag-Leffler Functions

2.1 A Few Biographical Notes On Gösta Magnus Mittag-Leffler

Gösta Magnus Mittag-Leffler was born on March 16, 1846, in Stockholm, Sweden. His father, John Olof Leffler, was a school teacher, and was also elected as a member of the Swedish Parliament. His mother, Gustava Vilhelmina Mittag, was a daughter of a pastor, who was a person of great scientific abilities. At his birth Gösta was given the name Leffler and later (when he was a student) he added his mother's name "Mittag" as a tribute to this family, which was very important in Sweden in the nineteenth century. Both sides of his family were of German origin.[1]

At the Gymnasium in Stockholm Gösta was training as an actuary but later changed to mathematics. He studied at the University of Uppsala, entering it in 1865. In 1872 he defended his thesis on applications of the argument principle and in the same year was appointed as a Docent (Associate Professor) at the University of Uppsala.

In the following year he was awarded a scholarship to study and work abroad as a researcher for three years. In October 1873 he left for Paris.

In Paris Mittag-Leffler met many mathematicians, such as Bouquet, Briot, Chasles, Darboux, and Liouville, but his main goal was to learn from Hermite. However, he found the lectures by Hermite on elliptic functions difficult to understand.

In Spring 1875 he moved to Berlin to attend lectures by Weierstrass, whose research and teaching style was very close to his own. From Weierstrass' lectures Mittag-Leffler learned many ideas and concepts which would later become the core of his scientific interests.

In Berlin Mittag-Leffler received news that professor Lorenz Lindelöf (Ernst Lindelöf's father) had decided to leave a chair at the University of Helsingfors (now Helsinki). At the same time Weierstrass requested from the ministry of education

[1] Many interesting aspects of Mittag-Leffler's life can be found in [Noe27, Har28a, Har28b].

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the installation of a new position at his institute and suggested Mittag-Leffler for the position. In spite of this, Mittag-Leffler applied for the chair at Helsingfors. He got the chair in 1876 and remained at the University of Helsingfors for the next five years.

In 1881 the new University of Stockholm was founded, and Gösta Mittag-Leffler was the first to hold a chair in Mathematics there. Soon afterwards he began to organize the setting up of the new international journal *Acta Mathematica*. In 1882 Mittag-Leffler founded *Acta Mathematica* and served as the Editor-in-Chief of the journal for 45 years. The original idea for such a journal came from Sophus Lie in 1881, but it was Mittag-Leffler's understanding of the European scene, together with his political skills, that ensured the success of the journal. Later he invited many well-known mathematicians (Cantor, Poincaré and many others) to submit papers to this journal. Mittag-Leffler was always a good judge of the quality of the work submitted to him for publication.

The role of G. Mittag-Leffler as a founder of Acta Mathematica was more than simply an organizer of the mathematical Journal. We can cite from [Dau80, p. 261–263]: "*Gösta Mittag-Leffler was the founding editor of the journal Acta Mathematica. In the early 1870s it was meant, in part, to bring the mathematicians of Germany and France together in the aftermath of the France-Prussian War, and the political neutrality of Sweden made it possible for Mittag-Leffler to realize this goal by publishing articles in German and French, side by side. Even before the end of the First World War, Mittag-Leffler again saw his role as mediator, and began to work for a reconciliation between German and Allied mathematicians through the auspices of his journal. Similarly, G.H. Hardy was particularly concerned about the reluctance of many scientists in England to attempt any sort of rapprochement with the Central European countries and he sought to do all he could to bring English and German mathematicians together after the War.... Nearly half a century earlier, Mittag-Leffler saw himself in much the same position as mediator between belligerent mathematicians on both sides. In fact, he believed that he was in an especially suitable position to bring European scientific interests together after World War I, and he saw his Acta Mathematics as the perfect instrument for promoting a lasting rapprochement.* "

In 1882 Gösta Mittag-Leffler married Signe af Linfors and they lived together until the end of his life.

Mittag-Leffler made numerous contributions to mathematical analysis, particularly in the areas concerned with limits, including calculus, analytic geometry and probability theory. He worked on the general theory of functions, studying relationships between independent and dependent variables.

His best known work deals with the analytic representation of a single-valued complex function, culminating in the Mittag-Leffler theorem. This study began as an attempt to generalize results from Weierstrass's lectures, where Weierstrass had described his theorem on the existence of an entire function with prescribed zeros each with a specified multiplicity. Mittag-Leffler tried to generalize this result to meromorphic functions while he was studying in Berlin. He eventually assembled his findings on generalizing Weierstrass' theorem to meromorphic functions in a paper which he published (in French) in 1884 in *Acta Mathematica*. In this paper

Mittag-Leffler proposed a series of general topological notions on infinite point sets based on Cantor's new set theory.

With this paper Mittag-Leffler became the sole proprietor of a theorem that later became widely known and so he took his place in the circle of internationally known mathematicians. Mittag-Leffler was one of the first mathematicians to support Cantor's theory of sets but, one has to remark, a consequence of this was that Kronecker refused to publish in *Acta Mathematica*. Between 1899 and 1905 Mittag-Leffler published a series of papers which he called "Notes" on the summation of divergent series. The aim of these notes was to construct the analytical continuation of a power series outside its circle of convergence. The region in which he was able to do this is now called Mittag-Leffler's star. Andre Weyl in his memorial [Weil82] says: "*A well-known anecdote has Oscar Wilde saying that he had put his genius into his life; into his writings he had put merely his talent. With at least equal justice it may be said of Mittag-Leffler that the Acta Mathematica were the product of his genius, while nothing more than talent went into his mathematical contributions. Genius transcends and defies analysis; but this may be a fitting occasion for examining some of the qualities involved in the creating and in the editing of a great mathematical journal.*"

In the same period Mittag-Leffler introduced and investigated in five subsequent papers a new special function, which is now very popular and useful for many applications. This function, as well as many of its generalizations, is now called the "Mittag-Leffler" function.[2]

His contribution is nicely summed up by Hardy [Har28a]: "*Mittag-Leffler was a remarkable man in many ways. He was a mathematician of the front rank, whose contributions to analysis had become classical, and had played a great part in the inspiration of later research; he was a man of strong personality, fired by an intense devotion to his chosen study; and he had the persistence, the position, and the means to make his enthusiasm count.*"

Gösta Mittag-Leffler passed away on July 7, 1927. During his life he received many honours. He was an honorary member or corresponding member of almost every mathematical society in the world including the Accademia Reale dei Lincei, the Cambridge Philosophical Society, the Finnish Academy of Sciences, the London Mathematical Society, the Moscow Mathematical Society, the Netherlands Academy of Sciences, the St. Petersburg Imperial Academy, the Royal Institution, the Royal Belgium Academy of Sciences and Arts, the Royal Irish Academy, the Swedish Academy of Sciences, and the Institute of France. He was elected a Fellow of the Royal Society of London in 1896. He was awarded honorary degrees from the Universities of Oxford, Cambridge, Aberdeen, St. Andrews, Bologna and Christiania (now Oslo).

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[2] Since it is the subject of this book, we will give below a wider discussion of these five papers and of the role of the Mittag-Leffler functions.

2.2 The Contents of the Five Papers by Mittag-Leffler on New Functions

Let us begin with a description of the ideas which led to the introduction by Mittag-Leffler of a new transcendental function.

In 1899 Mittag-Leffler began the publication of a series of articles under the common title "*Sur la représentation analytique d'une branche uniforme d'une fonction monogène*" ("On the analytic representation of a single-valued branch of a monogenic function") published mainly in *Acta Mathematica* [ML5-1, ML5-2, ML5-3, ML5-4, ML5-5, ML5-6]. The first articles of this series were based on three reports presented by him in 1898 at the Swedish Academy of Sciences in Stockholm.

His research was connected with the following question:
Let $k_0, k_1, \ldots$ be a sequence of complex numbers for which

$$\lim_{\nu\to\infty} |k_\nu|^{1/\nu} = \frac{1}{r} \in \mathbb{R}_+$$

is finite. Then the series

$$FC(z) := k_0 + k_1 z + k_2 z^2 + \ldots$$

is convergent in the disk $D_r = \{z \in \mathbb{C} : |z| < r\}$ and divergent at any point with $|z| > r$. It determines a single-valued analytic function in the disk D_r.[3]

The questions discussed were:

(1) to determine the maximal domain on which the function $FC(z)$ possesses a single-valued analytic continuation;
(2) to find an analytic representation of the corresponding single-valued branch.

Abel [Abe26a] had proposed (see also [Lev56]) to associate with the function $FC(z)$ the entire function

$$F_1(z) := k_0 + \frac{k_1 z}{1!} + \frac{k_2 z^2}{2!} + \ldots + \frac{k_\nu z^\nu}{\nu!} + \ldots = \sum_{\nu=0}^{\infty} \frac{k_\nu z^\nu}{\nu!}.$$

This function was used by Borel (see, e.g., [Bor01]) to discover that the answer to the above question is closely related to the properties of the following integral (now called the *Laplace–Abel integral*):

$$\int_0^\infty e^{-\omega} F_1(\omega z) d\omega. \qquad (2.2.1)$$

[3]The notation $FC(z)$ is not defined in Mittag-Leffler's paper. The letter "C" probably indicates the word 'convergent' in order to distinguish this function from its analytic continuation $FA(z)$ (see discussion below).

An intensive study of these properties was carried out at the beginning of the twentieth century by many mathematicians (see, e.g., [ML5-3, ML5-5] and references therein).

Mittag-Leffler introduced instead of $F_1(z)$ a one-parametric family of (entire) functions

$$F_\alpha(z) := k_0 + \frac{k_1 z}{\Gamma(1\cdot\alpha+1)} + \frac{k_2 z^2}{\Gamma(2\cdot\alpha+1)} + \ldots = \sum_{\nu=0}^{\infty} \frac{k_\nu z^\nu}{\Gamma(\nu\cdot\alpha+1)}, \ (\alpha > 0),$$

and studied its properties as well as the properties of the generalized Laplace–Abel integral

$$\int_0^\infty \mathrm{e}^{-\omega^{1/\alpha}} F_\alpha(\omega z)\mathrm{d}\omega^{1/\alpha} = \int_0^\infty \mathrm{e}^{-\omega} F_\alpha(\omega^\alpha z)\mathrm{d}\omega. \tag{2.2.2}$$

The main result of his study was: in a maximal domain A (star-like with respect to origin) the analytic representation of the single-valued analytic continuation $FA(z)$ of the function $FC(z)$ can be represented in the following form

$$FA(z) = \lim_{\alpha\to 1} \int_0^\infty \mathrm{e}^{-\omega} F_\alpha(\omega^\alpha z)\mathrm{d}\omega. \tag{2.2.3}$$

For this reason analytic properties of the functions $F_\alpha(z)$ become highly important.

Due to this construction Mittag-Leffler decided to study the most simple function of the type $F_\alpha(z)$, namely, the function corresponding to the unit sequence k_ν. This function

$$E_\alpha(z) := 1 + \frac{z}{\Gamma(1\cdot\alpha+1)} + \frac{z^2}{\Gamma(2\cdot\alpha+1)} + \ldots = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\nu\cdot\alpha+1)}, \tag{2.2.4}$$

was introduced and investigated by G. Mittag-Leffler in five subsequent papers [ML1, ML2, ML3, ML4, ML5-5] (in particular, in connection with the above formulated questions). This function is known now as the *Mittag-Leffler function*.

In the *first paper* [ML1], devoted to his new function, Mittag-Leffler discussed the relation of the function $F_\alpha(z)$ with the above problem on analytic continuation. In particular, he posed the question of whether the domains of analyticity of the function

$$\lim_{\alpha\downarrow 1} \int_0^\infty \mathrm{e}^{-\omega} F_\alpha(\omega^\alpha z)\mathrm{d}\omega$$

and the function (introduced and studied by Le Roy [LeR00]), see also [GaRoMa17, GoHoGa19] in relation to complete monotonicity of this function)

$$\lim_{\alpha\downarrow 1} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu+1)} k_\nu z^\nu$$

coincide.

In the *second paper* [ML2] the new function (i.e., the Mittag-Leffler function) appeared. Its asymptotic properties were formulated. In particular, Mittag-Leffler showed that $E_\alpha(z)$ behaves as $e^{z^{1/\alpha}}$ in the angle $-\frac{\pi\alpha}{2} < \arg z < \frac{\pi\alpha}{2}$ and is bounded for values of z with $\frac{\pi\alpha}{2} < |\arg z| \le \pi$.[4]

In the *third paper* [ML3] the asymptotic properties of $E_\alpha(z)$ were discussed more carefully. Mittag-Leffler compared his results with those of Malmquist [Mal03], Phragmén [Phr04] and Lindelöf [Lin03], which they obtained for similar functions (the results form the background of the classical Phragmén–Lindelöf theorem [PhrLin08]).

The *fourth paper* [ML4] was completely devoted to the extension of the function $E_\alpha(z)$ (as well as the function $F_\alpha(z)$) to complex values of the parameter α.

Mittag-Leffler's most creative paper on the new function $E_\alpha(z)$ is his *fifth paper* [ML5-5]. In this article, he:

(a) found an integral representation for the function $E_\alpha(z)$;
(b) described the asymptotic behavior of $E_\alpha(z)$ in different angle domains;
(c) gave the formulas connecting $E_\alpha(z)$ with known elementary functions;
(d) provided the asymptotic formulas for

$$E_\alpha(z) = \frac{1}{2\pi i}\int_L \frac{1}{\alpha} e^{\omega^{1/\alpha}} \frac{d\omega}{\omega - z}$$

by using the so-called *Hankel integration path*;
(e) obtained detailed asymptotics of $E_\alpha(z)$ for negative values of the variable, i.e. for $z = -r$;
(f) compared in detail his asymptotic results for $E_\alpha(z)$ with the results obtained by Malmquist;
(g) found domains which are free of zeros of $E_\alpha(z)$ in the case of "small" positive values of parameter, i.e. for $0 < \alpha < 2,\ \alpha \neq 1$;
(h) applied his results on $E_\alpha(z)$ to answer the question of the domain of analyticity of the function $FA(z)$ and its analytic representation (see formula (2.2.3)).

2.3 Further History of Mittag-Leffler Functions

The importance of the new function was understood as soon as the first analytic results for it appeared. First of all, it is a very simple function playing a key role in the solution of a general problem of the theory of analytic functions. Secondly, the Mittag-Leffler function can be considered as a direct generalization of the exponential function, preserving some of its properties. Furthermore, $E_\alpha(z)$ has some interesting properties which later became essential for the description of many problems arising in applications.

[4] The behavior of $E_\alpha(z)$ on critical rays $|\arg z| = \pm\frac{\pi\alpha}{2}$ was not described.

After Mittag-Leffler's introduction of the new function, one of the first results on it was obtained by Wiman [Wim05a]. He used Borel's method of summation of divergent series (which Borel applied to the special case of the Mittag-Leffler function, namely, for $\alpha = 1$, see [Bor01]). Using this method, Wiman gave a new proof of the asymptotic representation of $E_\alpha(z)$ in different angle domains. This representation was obtained for positive rational values of the parameter α. He also noted[5] that analogous asymptotic results hold for the two-parametric generalization $E_{\alpha,\beta}(z)$ of the Mittag-Leffler function (see (1.0.3)). Applying the obtained representation Wiman described in [Wim05b] the distribution of zeros of the Mittag-Leffler function $E_\alpha(z)$. The main focus was on two cases – to the case of real values of the parameter $\alpha \in (0, 2]$, $\alpha \neq 1$, and to the case of complex values of α, $\mathrm{Re}\,\alpha > 0$.

In [Phr04] Phragmén proved the generalization of the Maximum Modulus Principle for the case of functions analytic in an angle. For this general theorem the Mittag-Leffler function plays the role of the key example. It satisfies the inequality $|E_\alpha(z)| < C_1 e^{|z|^\rho}$, $\rho = 1/\{\mathrm{Re}\,\alpha\}$, in an angular domain z, $|\arg z| \le \frac{\pi}{2\rho}$, but although it is bounded on the boundary rays it is not constant in the whole angular domain. This means that the Mittag-Leffler function possesses a maximal angular domain (in the sense of the *Phragmén* or *Phragmén–Lindelöf theorem*, see [PhrLin08]) in which the above stated property holds.

One more paper devoted to the development of the asymptotic method of Mittag-Leffler appeared in 1905. Malmquist (a student of G. Mittag-Leffler) applied this method to obtain the asymptotics of a function similar to $E_\alpha(z)$, namely

$$\sum_{\nu} \frac{z^\nu}{\Gamma(1 + \nu a_\nu)},$$

where the sequence a_ν tends to zero as $\nu \to \infty$. The particular goal was to construct a simple example of an entire function which tends to zero along almost all rays when $|z| \to \infty$. Such an example

$$G(z) = \sum_{\nu} \frac{z^\nu}{\Gamma(1 + \frac{\nu}{(\log \nu)^\alpha})}, \quad 0 < \alpha < 1, \tag{2.3.1}$$

was constructed [Mal05] and carefully examined by using the calculus of residues for the integral representation of $G(z)$ (which is also analogous to that for E_α).

At the beginning of the twentieth century many mathematicians paid great attention to obtaining asymptotic expansions of special functions, in particular, those of hypergeometric type. The main reason for this was that these functions play an important role in the study of differential equations, which describe different phenomena. In the fundamental paper [Barn06] Barnes proposed a unified approach to the investigation of asymptotic expansions of entire functions defined by Taylor series. This

[5] But did not discuss in detail.

approach was based on the previous results of Barnes [Barn02] and Mellin [Mel02]. The essence of this approach is to use the representation of the quotient of the products of Gamma functions in the form of a contour integral which is handled by using the method of residues. This representation is now known as the *Mellin–Barnes integral formula* (see Appendix D). Among the functions which were treated in [Barn06] was the Mittag-Leffler function. The results of Barnes were further developed in his articles, including applications to the theory of differential equations, as well as in the articles by Mellin (see, e.g., [Mel10]). In fact, the idea of employing contour integrals involving Gamma functions of the variable in the subject of integration is due to Pincherle, whose suggestive paper [Pin88] was the starting point of Mellin's investigations (1895), although the type of contour and its use can be traced back to Riemann, as Barnes wrote in [Barn07b, p. 63].

Generalizations of the Mittag-Leffler function are proposed among other generalizations of the hypergeometric functions. For them similar approaches were used. Among these generalizations we should point out the collection of *Wright functions*, first introduced in 1935, see [Wri35a],

$$\phi(z;\rho,\beta) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(\rho n+\beta)} = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\rho n+\beta)}; \tag{2.3.2}$$

the collection of generalized hypergeometric functions, first introduced in 1928, see [Fox28],

$${}_pF_q(z) = {}_pF_q\left(\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q;z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdot(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdot(\beta_2)_k\cdots(\beta_q)_k}\frac{z^k}{k!}; \tag{2.3.3}$$

[6]the collection of *Meijer G-functions* introduced in 1936, see [Mei36], and intensively treated in 1946, see [Mei46],

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_T \frac{\prod_{i=1}^{m}\Gamma(b_i+s)\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=m+1}^{q}\Gamma(1-b_i-s)\prod_{i=n+1}^{p}\Gamma(a_i+s)} z^{-s}\mathrm{d}s, \tag{2.3.4}$$

[6]Here $(\cdot)_k$ is the Pochhammer symbol, see (A.17) in Appendix A.

and the collection of more general *Fox H-functions*

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_T \frac{\prod_{i=1}^{m}\Gamma(b_i+\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=m+1}^{q}\Gamma(1-b_i-\beta_i s)\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)} z^{-s}\mathrm{d}s. \tag{2.3.5}$$

Some generalizations of the Mittag-Leffler function appeared as a result of developments in integral transform theory. In this connection in 1953 Agarwal and Humbert (see [Hum53, Aga53, HumAga53]) and independently in 1954 Djrbashian (see [Dzh54a, Dzh54b, Dzh54c]) introduced and studied the *two-parametric Mittag-Leffler function* (or *Mittag-Leffler type function*)

$$E_{\alpha,\beta}(z) := \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\nu\cdot\alpha+\beta)}. \tag{2.3.6}$$

We note once more that, formally, the function (2.3.6) first appeared in the paper of Wiman [Wim05a], who did not pay much attention to its extended study.

In 1971 Prabhakar [Pra71] introduced the *three-parametric Mittag-Leffler function* (or *generalized Mittag-Leffler function*, or *Prabhakar function*)

$$E_{\alpha,\beta}^{\rho}(z) := \sum_{\nu=0}^{\infty} \frac{(\rho)_\nu z^\nu}{\Gamma(\nu\cdot\alpha+\beta)}. \tag{2.3.7}$$

This function appeared in the kernel of a first-order integral equation which Prabhakar treated by using Fractional Calculus.

Other *three-parametric Mittag-Leffler functions* (also called *generalized Mittag-Leffler functions* or *Mittag-Leffler type functions*, or *Kilbas–Saigo functions*) were introduced by Kilbas and Saigo (see, e.g., [KilSai95a])

$$E_{\alpha,m,l}(z) := \sum_{n=0}^{\infty} c_n z^n, \tag{2.3.8}$$

where

$$c_0 = 1,\ \ c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(im+l)+1]}{\Gamma[\alpha(im+l+1)+1]}.$$

These functions appeared in connection with the solution of new types of integral and differential equations and with the development of the Fractional Calculus. Nowadays they are referred to as Kilbas–Saigo functions.