Frontiers in Mathematics

Rainer Picard · Des McGhee
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A Primer for a Secret Shortcut to PDEs of Mathematical Physics

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Introduction

A typical entry point into the field of (linear) partial differential equations is to consider general polynomials $P(\partial)$ in $\partial := (\partial_0, \ldots, \partial_n)$ with (complex or real) matrix coefficients. Here ∂_k denotes the partial derivative with respect to the variable in the position labelled with[1] $k \in \{0, \ldots, n\}$, $n \in \mathbb{N}$. Even if we discuss solutions only in the whole Euclidean space $\mathbb{R}^{n+1}$ the solution theory for an equation of the form

$$P(\partial)\,u = f$$

involving a general partial differential operator $P(\partial)$ is quite involved and one quickly restricts attention to very specific polynomials. Indeed, the equations relevant to applications are not that varied. One commonly investigates three subclasses, loosely labelled as elliptic, parabolic, and hyperbolic, to present specific solution methods for each of them.

However, when viewed from the right perspective there is a single subclass containing these three types (and many more), which can be characterized conveniently and solved with one and the same method. To explain the corresponding rigorous framework is the objective of this text.

The theory we will present in this book is rooted in [57], with some first generalizations to be found in [55,59]. We shall refer also to [63,79,83,87,88] for generalizations towards nonlinear or non-autonomous setups. The interested reader will find a more detailed survey in [68,75]. In the present book, however, we shall present the core yet surprisingly elementary solution theory for what we will call *evolutionary equations*.

The structure of this class of partial differential expressions can be formally described by two matrices[2] $M_0, M_1 \in \mathbb{R}^{(N+1)\times(N+1)}$, $N \in \mathbb{N}$. The partial differential operator

[1] Note that we usually prefer to start our numbering with 0. In particular, $\mathbb{N}$ denotes the set of non-negative integers.

[2] Indeed, keeping in mind that a complex number $x + \mathrm{i}y$ can be understood as a (2×2)-matrix of the form

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

where $x, y \in \mathbb{R}$, we may actually assume that M_0 and M_1 have only real entries.

$P(\partial)$ will then be assumed to be of the form

$$P(\partial) = \partial_0 M_0 + M_1 + A\left(\widehat{\partial}\right), \tag{1}$$

where $A\left(\widehat{\partial}\right)$ denotes a polynomial in $\widehat{\partial} := (\partial_1, \ldots, \partial_n)$, that is, borrowing jargon from applied fields, only in the "spatial" variables, if we consider ∂_0 to be the derivative with respect to "time". In this terminology, if we focus on "relevant" partial differential equations, we may focus on first-order-in-time systems. Moreover, in standard cases we have structural features of $P(\partial)$ which narrow down the class of differential operators even further. We assume[3]

$$A^*\left(-\widehat{\partial}\right) = -A\left(\widehat{\partial}\right)$$

and

$$M_0 = M_0^* \text{ and } \mathfrak{Re}\, M_1 := \frac{1}{2}\left(M_1 + M_1^*\right) \geq c_0 > 0. \tag{2}$$

In applications, the latter positive definiteness constraint is rarely satisfied. However, after a simple formal transformation[4] we get

$$\partial_0 M_0 + \widetilde{M}_1 + A\left(\widehat{\partial}\right) = \exp\left(-\varrho m_0\right)\left(\partial_0 M_0 + M_1 + A\left(\widehat{\partial}\right)\right)\exp\left(\varrho m_0\right)$$

---

[3]With this, $A\left(\widehat{\partial}\right)$ becomes skew-selfadjoint in $L^2\left(\mathbb{R}^n\right)$ and—by canonical extension to the time-dependent case—in $L^2\left(\mathbb{R}^{1+n}\right)$. If $A\left(\widehat{\partial}\right) = \sum_{\alpha\in\mathbb{N}^n} A_\alpha \widehat{\partial}^\alpha$, then $A\left(\widehat{\partial}\right)^* = A^*\left(-\widehat{\partial}\right) := \sum_{\alpha\in\mathbb{N}^n} A_\alpha^*\left(-\widehat{\partial}\right)^\alpha$ and this constraint means that the matrix coefficients A_α, $\alpha = (\alpha_1, \ldots, \alpha_n)$, are selfadjoint or skew-selfadjoint depending on the order $|\alpha| := \sum_{k=1}^n \alpha_k$ being even or odd, respectively. Note that since $A\left(\widehat{\partial}\right)$ is a polynomial, only finitely many of the coefficients are non-vanishing. In most cases, the maximal order is actually also just 1.

[4]This transformation shifts the rigorous functional analytical discussion from $L^2\left(\mathbb{R}^{n+1}\right)$ to the more appropriate setting in the Hilbert space $H_{\varrho,0}\left(\mathbb{R}; L^2\left(\mathbb{R}^n\right)\right)$, which is defined such that

$$\exp\left(-\varrho m_0\right) : H_{\varrho,0}\left(\mathbb{R}; L^2\left(\mathbb{R}^n\right)\right) \to L^2\left(\mathbb{R}; L^2\left(\mathbb{R}^n\right)\right) = L^2\left(\mathbb{R}^{1+n}\right)$$
$$\varphi \mapsto \exp\left(-\varrho m_0\right)\varphi$$

becomes a unitary mapping. Here the multiplication operator $\exp\left(-\varrho m_0\right)$ is defined via $\left(\exp\left(-\varrho m_0\right)\varphi\right)(t) := \exp\left(-\varrho t\right)\varphi(t)$, $t \in \mathbb{R}$. We will be more precise and detailed later.

where

$$\widetilde{M}_1 := M_1 - \varrho M_0.$$

Now, the constraints (2) translate to

$$M_0 = M_0^* \text{ and } \varrho M_0 + \mathfrak{Re}\, M_1 \geq c_0 > 0 \tag{3}$$

and the latter strict positive definiteness constraint needs to hold only for all sufficiently large $\varrho \in\,]0, \infty[$. As we shall see later, the particular role of time is encoded in this bias for *positive* values of parameter ϱ.

To improve on the range of applicability, we will generalize the above problem class by allowing M_0 and M_1 to be Hilbert space operators and A to be a general skew-selfadjoint operator so that operators of the "space-time" form

$$\partial_0 M_0 + M_1 + A \tag{4}$$

can be treated. In the proper setting, ∂_0 will be seen to be a continuously invertible operator, which, among other things, allows us to consider the operator $M\left(\partial_0^{-1}\right) := M_0 + \partial_0^{-1} M_1$, which in application occurs when describing so-called material laws. We therefore shall refer to $M\left(\partial_0^{-1}\right)$ as well as to M_0 and M_1 as material law operators. This setting essentially yields a new *normal form* for partial differential equations occurring in numerous applications.

In the following, we shall rigorously develop the solution theory of these abstract equations, which—due to their implied causality properties—we refer to as evolutionary equations. We use the term *evolutionary* in a somewhat subtle attempt to distinguish them from the classical concept of *evolution equations*, which are explicit first-order-in-time equations.

Although this class can be readily generalized to include more complicated cases, such as merely assuming that the numerical ranges of A, A^* are in the closed complex right half-plane or allowing for more complicated material law operators $M\left(\partial_0^{-1}\right)$ with the positive definiteness constraint that for some $c_0 \in\,]0, \infty[$ the numerical range of $\partial_0 M\left(\partial_0^{-1}\right) - c_0$ is in the closed complex right half-plane (for all sufficiently large $\varrho \in\,]0, \infty[$) (see again e.g. [59, 75]), we shall focus here on the more easily accessible pure differential case.

Eventually, we aim at a solution theory with easy to check assumptions that lead to well-posedness of a rather large class of partial differential equations. Indeed, we will see that well-posedness of an evolutionary equation boils down to proving a numerical range constraint for certain bounded operators only.

In Chap. 1, we develop the functional analytical setting and the basic solution theory. Chapter 2 illustrates the theory for a number of model problems from mathematical physics. This concludes the book's core material. Chapter 3 addresses some of the issues that may arise when comparing our approach with some alternative, possibly more mainstream ideas for dealing with problems of the same type. Two appendices complement the book's material by providing additional ideas for expanding on the applicability of the approach, Appendix A, and collecting some background material from functional analysis as a study resource, Appendix B.

Contents

1 The Solution Theory for a Basic Class of Evolutionary Equations

1.1 The Time Derivative

We start out with the definition of the time derivative. We emphasize that all vector spaces discussed in this exposition have the real numbers as underlying scalar field. This is a simplifying assumption. If given a complex Hilbert space, restrict the underlying scalar field to real multipliers and the scalar product to its real part. In this way the results developed here apply to the complex Hilbert space case as well. Note that, however, this reasoning can also be dispensed with and the complex Hilbert space case may be addresses directly, see the original work [58] for this. The exposition roughly follows [88, Chapter 1].

Definition 1.1.1 (Time Derivative) Let $L^2(\mathbb{R})$ be the Hilbert space of (equivalence classes of) square-integrable real-valued functions on $\mathbb{R}$. For $\varrho \in \mathbb{R}$ we define $H_{\varrho,0}(\mathbb{R}) := \{f \in L^2_{\mathrm{loc}}(\mathbb{R});\ (t \mapsto \exp(-\varrho t) f(t)) \in L^2(\mathbb{R})\}$ as a Hilbert space equipped with the inner product

$$\langle u|v\rangle_{\varrho,0} := \int_{\mathbb{R}} u(t)v(t)\exp(-2\varrho t)\,dt \quad (u, v \in H_{\varrho,0}(\mathbb{R})).$$

We set

$$\begin{aligned}\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})} : \mathring{C}_1(\mathbb{R}) \subseteq H_{\varrho,0}(\mathbb{R}) &\to H_{\varrho,0}(\mathbb{R}),\\ u &\mapsto u',\end{aligned}$$

where $\mathring{C}_1(\mathbb{R})$ is the space of compactly supported continuously differentiable functions.

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Clearly, for all $\varrho \in \mathbb{R}$, the operator $\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}$ is densely defined. The operator is also closable:

Proposition 1.1.2 *For all $v \in \mathring{C}_1(\mathbb{R})$ we have*

$$\left(\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}\right)^* v = -v' + 2\varrho v,$$

hence $\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}$ is closable.

Proof We note here that it suffices to prove the asserted equality. For if the equality is true, the adjoint of $\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}$ is densely defined and thus $\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}$ is closable. So, let $u, v \in \mathring{C}_1(\mathbb{R})$. Then we compute with the help of integration by parts

$$\begin{aligned}
\langle \partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})} u | v\rangle_{\varrho,0} &= \langle u' | v\rangle_{\varrho,0} \\
&= \int_{\mathbb{R}} u'(t)v(t)\exp(-2\varrho t)dt \\
&= -\int_{\mathbb{R}} \big(u(t)v'(t)\exp(-2\varrho t) - 2\varrho u(t)v(t)\exp(-2\varrho t)\big)\,dt \\
&= \langle u | -v'\rangle_{\varrho,0} + \langle u | 2\varrho v\rangle_{\varrho,0}.
\end{aligned}$$

This yields the assertion. □

We define

$$\partial_{0,\varrho} := \overline{\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}}.$$

A consequence of the latter proposition is

$$-\partial_{0,\varrho} + 2\varrho \subseteq \partial_{0,\varrho}^*. \tag{1.1.1}$$

Among other things we will show in the following that here equality is true. The strategy of the proof is to consider the inverse of $\partial_{0,\varrho}$ first. We define

$$L^1_\varrho(\mathbb{R}) := \{h \in L^1_{\mathrm{loc}}(\mathbb{R});\, (t \mapsto \exp(-\varrho t)h(t)) \in L^1(\mathbb{R})\}$$

for all $\varrho \in \mathbb{R}$ and recall Young's inequality.

Proposition 1.1.3 (Young's Inequality) *Let* $\varrho \in \mathbb{R}$, $h \in L^1_\varrho(\mathbb{R})$, $f \in \mathring{C}_1(\mathbb{R})$. *Then for all* $t \in \mathbb{R}$

$$h * f(t) := \int_{\mathbb{R}} h(t-s) f(s) ds$$

is well-defined and $t \mapsto h * f(t) \in H_{\varrho,0}(\mathbb{R})$ *with*

$$|h * f|_{\varrho,0} \leq |h|_{L^1_\varrho} |f|_{\varrho,0}$$

holds. In particular, $h*$ *extends to a bounded linear operator on* $H_{\varrho,0}(\mathbb{R})$ *with* $\|h*\| \leq |h|_{L^1_\varrho}$.

Proof Note that by a change of variables,

$$h * f(t) = \int_{\mathbb{R}} h(s) f(t-s) ds$$

for all $t \in \mathbb{R}$. This implies the existence of the integral (and even the continuity of $h * f$ by Lebesgue's dominated convergence theorem). Next, we estimate using the Cauchy–Schwarz inequality

$$\begin{aligned}
\|h * f\|^2_{\varrho,0} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} h(t-s) f(s) ds \right|^2 \exp(-2\varrho t) dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t-s)| \exp(-\varrho(t-s)) \, |f(s)| \exp(-\varrho s) ds \right)^2 dt \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (|h(t-s)| \exp(-\varrho(t-s)))^{1/2+1/2} \, |f(s)| \exp(-\varrho s) ds \right)^2 dt \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t-s)| \exp(-\varrho(t-s)) ds \times \right. \\
&\qquad \left. \times \int_{\mathbb{R}} \left|h(t-s')\right| \exp(-\varrho(t-s')) \left|f(s')\right|^2 \exp(-2\varrho s') ds' \right) dt \\
&= |h|_{L^1_\varrho} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|h(t-s')\right| \exp(-\varrho(t-s')) dt \left|f(s')\right|^2 \exp(-2\varrho s') ds' \\
&= |h|^2_{L^1_\varrho} |f|^2_{\varrho,0},
\end{aligned}$$

yielding the assertion. □

A particular application of the latter estimate concerns the following two cases

$$h = \chi_{[0,\infty[} \in L^1_\varrho(\mathbb{R}) \tag{1.1.2}$$

and

$$h = -\chi_{]-\infty,0]} \in L^1_{-\varrho}(\mathbb{R}) \tag{1.1.3}$$

for all $\varrho > 0$. Note that in either case, we have $|h|_{L^1_\varrho} = 1/|\varrho|$ for $\varrho \neq 0$. Moreover, it is easy to see that $\chi_{[t,\infty)}(\cdot)f(\cdot) \in L^1_\varrho(\mathbb{R})$ for all $f \in H_{\varrho,0}(\mathbb{R})$, $t \in \mathbb{R}$, $\varrho > 0$, so that

$$t \mapsto h * f(t) = \int_{-\infty}^{t} f(s)ds$$

is well-defined and continuous (and analogously for $\varrho < 0$).

With these settings at hand, we prove the bounded invertibility of $\partial_{0,\varrho}$, $\varrho \neq 0$:

Theorem 1.1.4 *Let $\varrho \neq 0$. Then the operator $\partial_{0,\varrho}$ is continuously invertible, $\partial_{0,\varrho}^{-1} = h*$ with h respective of the sign of ϱ as in* (1.1.2) *or* (1.1.3) *and*

$$\|\partial_{0,\varrho}^{-1}\| \leq \frac{1}{|\varrho|}.$$

Proof We only prove the case $\varrho > 0$, the case $\varrho < 0$ being analogous. Let $f \in \mathring{C}_1(\mathbb{R})$ and let $\varphi \in \mathring{C}_1(\mathbb{R})$ be such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $[-1, 1]$. For $n \in \mathbb{N}_{>0}$ we denote $\varphi_n := \varphi\left(\frac{\cdot}{n}\right)$. Then, by the fundamental theorem of calculus, we get

$$\begin{aligned}
\partial_{0,\varrho}(\varphi_n(h * f)) &= \partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R})}(\varphi_n(h * f)) \\
&= (\varphi_n(h * f))' \\
&= \varphi_n'(h * f) + \varphi_n f \\
&= \frac{1}{n}\varphi'\left(\frac{\cdot}{n}\right)(h * f) + \varphi_n f.
\end{aligned}$$

Letting $n \to \infty$, we deduce $h * f \in \operatorname{dom}(\partial_{0,\varrho})$ and $\partial_{0,\varrho}(h * f) = f$. Indeed, this follows from $\varphi_n(h * f) \to h * f$ and $\frac{1}{n}\varphi'\left(\frac{\cdot}{n}\right)(h * f) + \varphi_n f \to f$ in $H_{\varrho,0}(\mathbb{R})$ and the closedness of $\partial_{0,\varrho}$. Next, for $f \in H_{\varrho,0}(\mathbb{R})$ there exists a sequence $(f_n)_n$ in $\mathring{C}_1(\mathbb{R})$ such that $f_n \to f$ in $H_{\varrho,0}(\mathbb{R})$. By Proposition 1.1.3, we deduce that $h * f_n \to h * f$ in $H_{\varrho,0}(\mathbb{R})$. And so, from $\partial_{0,\varrho}h * f_n = f_n$ we deduce that $h * f \in \operatorname{dom}(\partial_{0,\varrho})$ and $\partial_{0,\varrho}(h * f) = f$.

Next, let $f \in \operatorname{dom}(\partial_{0,\varrho})$ and $g := \partial_{0,\varrho}f$. There exists a sequence $(f_n)_n$ in $\mathring{C}_1(\mathbb{R})$ with the property that $f_n \to f$ and $g_n := \partial_{0,\varrho}f_n = f_n' \to g$ as $n \to \infty$ in $H_{\varrho,0}(\mathbb{R})$, by definition of $\partial_{0,\varrho}$. Thus, by Proposition 1.1.3 and the fundamental theorem of calculus

$$\begin{aligned} h * \partial_{0,\varrho} f &= h * g \\ &= \lim_{n\to\infty} h * g_n \\ &= \lim_{n\to\infty} h * f_n' \\ &= \lim_{n\to\infty} f_n = f, \end{aligned}$$

which yields the assertion. □

Corollary 1.1.5 *Let $\varrho \in \mathbb{R}$. Then*

$$\partial_{0,\varrho}^* = -\partial_{0,\varrho} + 2\varrho.$$

Proof Consider the unitary mapping

$$\begin{aligned} \exp(-\varrho m) : H_{\varrho,0}(\mathbb{R}) &\to L^2(\mathbb{R}) \\ u &\mapsto (t \mapsto \exp(-\varrho t)u(t)) \end{aligned}$$

and its adjoint/inverse

$$\begin{aligned} \exp(-\varrho m)^* : L^2(\mathbb{R}) &\to H_{\varrho,0}(\mathbb{R}) \\ v &\mapsto (t \mapsto \exp(\varrho t)v(t))\,. \end{aligned}$$

Then an easy computation shows

$$\partial_{0,\varrho} = \exp(-\varrho m)^* \left(\partial_{0,0} + \varrho\right) \exp(-\varrho m). \tag{1.1.4}$$

Indeed, the result is clear for elements in $\mathring{C}_1(\mathbb{R})$ and by taking closures, the equality follows. In particular, we see that the operators

$$\partial_{0,0} \pm 1$$

are boundedly invertible on $L^2(\mathbb{R})$ since both are unitarily equivalent to the invertible operators $\partial_{0,\pm 1}$, respectively. Since by (1.1.1) we have that

$$\partial_{0,0} + 1 \subseteq -\partial_{0,0}^* + 1 = -(\partial_{0,0} - 1)^*$$

we derive equality, because the operator on the left-hand side is onto and the operator on the right-hand side is one-to-one (using Theorem B.4.8). Summarizing, we have shown

$$\partial_{0,0} = -\partial_{0,0}^*.$$

According to (1.1.4) this, however, implies

$$\begin{aligned}\partial_{0,\varrho}^* &= \exp(-\varrho m)^* \left(\partial_{0,0} + \varrho\right)^* \exp(-\varrho m)\\ &= \exp(-\varrho m)^* \left(-\partial_{0,0} + \varrho\right) \exp(-\varrho m)\\ &= -\partial_{0,\varrho} + 2\varrho,\end{aligned}$$

which shows the claim. □

We remark here that another consequence of the equality in Corollary 1.1.5 is that $\mathring{C}_1(\mathbb{R})$ is an operator core not only for $\partial_{0,\varrho}$ but also for $\partial_{0,\varrho}^*$. Moreover, we obtain that $\operatorname{dom}(\partial_{0,\varrho}) = \operatorname{dom}(\partial_{0,\varrho}^*)$. Note that the results obtained in this section carry over almost verbatim to the case of H-valued $H_{\varrho,0}$-functions, that is, to functions in the space

$$H_{\varrho,0}(\mathbb{R}; H) := \{f \in L^2_{\text{loc}}(\mathbb{R}; H); (t \mapsto \exp(-\varrho t) f(t)) \in L^2(\mathbb{R}; H)\}.$$

We summarize this in the following theorem, which for simplicity we only formulate for the case $\varrho > 0$.

Theorem 1.1.6 *Let $\varrho \in \mathbb{R}_{>0}$, and let H be a Hilbert space. Define*

$$\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R};H)} \colon \mathring{C}_1(\mathbb{R}; H) \subseteq H_{\varrho,0}(\mathbb{R}; H) \to H_{\varrho,0}(\mathbb{R}; H), \varphi \mapsto \varphi'.$$

Then $\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R};H)}$ is densely defined and closable; $\partial_{0,\varrho} := \overline{\partial_{0,\varrho}|_{\mathring{C}_1(\mathbb{R};H)}}$ is continuously invertible and for all $f \in H_{\varrho,0}(\mathbb{R}; H)$ we have

$$\partial_{0,\varrho}^{-1} f(t) = \int_{-\infty}^t f(s)\, ds \quad (t \in \mathbb{R}).$$

Furthermore, $\|\partial_{0,\varrho}^{-1}\| \le 1/|\varrho|$ and $\partial_{0,\varrho}^* = -\partial_{0,\varrho} + 2\varrho$.

For $\varrho > 0$, the formula for the inverse of $\partial_{0,\varrho}$ reveals that the solution u of the equation $\partial_{0,\varrho} u = f$ up to a certain time $t \in \mathbb{R}$ does not depend on the behavior of f from $t \in \mathbb{R}$ onwards. This property is called causality and will be described by means of an estimate in the following theorem. The additional linear operator $M_0 \in \mathcal{B}(H)$ mentioned in the following statement can be thought of being the identity operator on H on a first read.

Moreover, when applied to elements in $H_{\varrho,0}(\mathbb{R};H)$, the operator M_0 is to be understood in the point-wise sense, that is, $(M_0 u)(t) := M_0(u(t))$ for each $u \in H_{\varrho,0}(\mathbb{R};H)$. We note that the inequality will play a crucial role in our analysis of (evolutionary) partial differential equations to follow.

Theorem 1.1.7 *Let $\varrho \in \mathbb{R}_{>0}$, let H be a Hilbert space, and let $0 \le M_0 = M_0^* \in L(H)$. Then for all $u \in \operatorname{dom}(\partial_{0,\varrho})$ and $a \in \mathbb{R}$ we have*

$$\langle \partial_{0,\varrho} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0} \ge \varrho \langle \chi_{]-\infty,a]} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0}.$$

Proof By Theorem 1.1.6, it suffices to prove the inequality for $u \in \mathring{C}_1(\mathbb{R};H)$. We compute for $a \in \mathbb{R}$ using integration by parts and the fact that $M_0 u \in \mathring{C}_1(\mathbb{R};H)$, by the linearity, boundedness and selfadjointness of M_0,

$$\begin{aligned}
&\langle \partial_{0,\varrho} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0} \\
&= \int_{-\infty}^{a} \langle (M_0 u)'(s) | u(s)\rangle \exp(-2\varrho s) ds \\
&= -\int_{-\infty}^{a} \langle M_0 u(s) | u'(s)\rangle \exp(-2\varrho s) ds \\
&\qquad + 2\varrho \int_{-\infty}^{a} \langle M_0 u(s) | u(s)\rangle \exp(-2\varrho s) ds + \langle M_0 u(a) | u(a)\rangle \exp(-2\varrho a) \\
&\ge -\int_{-\infty}^{a} \langle u(s) | (M_0 u)'(s)\rangle \exp(-2\varrho s) ds + 2\varrho \int_{-\infty}^{a} \langle M_0 u(s) | u(s)\rangle \exp(-2\varrho s) ds.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\langle \partial_{0,\varrho} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0} + \langle \chi_{]-\infty,a]} u | \partial_{0,\varrho} M_0 u\rangle_{\varrho,0} \\
&\qquad = 2\langle \partial_{0,\varrho} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0} \ge 2\varrho \langle \chi_{]-\infty,a]} M_0 u | \chi_{]-\infty,a]} u\rangle_{\varrho,0}. \qquad \square
\end{aligned}$$

To simplify notation we shall write ∂_0 instead of $\partial_{0,\varrho}$ if ϱ is clear from the context. Although in the one-dimensional case the index 0 is not really needed, we use this notation to underscore that ∂_0 will serve as our realization of the time derivative. (We anticipate the introduction of 'spatial' derivatives for which we shall use the indices starting with 1.)

A particular instance of Theorem 1.1.7 is $M_0 = 1$: Then we have

$$\left\langle \chi_{]-\infty,a]} u | \partial_0 u\right\rangle_{\varrho,0} = \left\langle u | \chi_{]-\infty,a]} \partial_0 u\right\rangle_{\varrho,0} \ge \varrho_0 \left\langle \chi_{]-\infty,a]} u | \chi_{]-\infty,a]} u\right\rangle_{\varrho,0}$$

for all $a \in \mathbb{R}$ and all $\varrho \in [\varrho_0, \infty[$, which precisely underpins the property of causality mentioned above: if $f = \partial_0 u$ vanishes on an interval $]-\infty, a]$ then so does $\partial_0^{-1} f = u$.

This property can also be expressed in the form

$$\chi_{]-\infty,a]}\partial_0^{-1}\left(1-\chi_{]-\infty,a]}\right)=0$$

or

$$\chi_{]-\infty,a]}\partial_0^{-1}=\chi_{]-\infty,a]}\partial_0^{-1}\chi_{]-\infty,a]}$$

for all $a\in\mathbb{R}$. Before we turn to partial differential equations, we will consider the derivative just defined in the context of ordinary differential equations, see also [22, 75]. The following corollary however, while involving only the one derivative, is essential for our analysis of partial differential equations.

Corollary 1.1.8 *Let H Hilbert space, $\varrho,\varepsilon>0$. Then both $1+\varepsilon\partial_0$ and $1+\varepsilon\partial_0^*$ are continuously invertible. The operator norm of the inverses are bounded by 1 and*

$$(1+\varepsilon\partial_0)^{-1},\left((1+\varepsilon\partial_0)^{-1}\right)^*=\left(1+\varepsilon\partial_0^*\right)^{-1}\to 1_{H_{\varrho,0}(\mathbb{R};H)}$$

in the strong operator topology as $\varepsilon\to 0$.

Proof Let $u\in\operatorname{dom}(\partial_0)=\operatorname{dom}(\partial_0^*)$ (see Theorem 1.1.6). We compute with the help of Theorem 1.1.7:

$$\langle(1+\varepsilon\partial_0)\,u|u\rangle_{\varrho,0}=\langle u|\left(1+\varepsilon\partial_0^*\right)u\rangle_{\varrho,0}\geq\langle u|u\rangle_{\varrho,0}+\varepsilon\varrho\langle u|u\rangle_{\varrho,0}\geq\langle u|u\rangle_{\varrho,0}.$$

Furthermore, from $(1+\varepsilon\partial_0)^{-1}u=u-\varepsilon(1+\varepsilon\partial_0)^{-1}\partial_0 u\to u$ as $\varepsilon\to 0$ for all $u\in\operatorname{dom}(\partial_0)$ and from $\sup_{\varepsilon>0}\|(1+\varepsilon\partial_0)^{-1}\|\leq 1$, we deduce the first convergence statement. The second one is similar. □

Remark 1.1.9 This corollary is a special case of Lemma B.7.1. Indeed, it suffices to observe that causality of $\partial_{0,\varrho}$ (in the form of Theorem 1.1.7) particularly implies the accretivity of $\partial_{0,\varrho}$ and of its adjoint $\partial_{0,\varrho}^*$.

1.2 A Hilbert Space Perspective on Ordinary Differential Equations

The above discussion suggests a Hilbert space theory for ordinary differential equations, which we explore for a moment. A more detailed exposition can be found in [22, 75] for the Hilbert space and [64] for the Banach space case.

Indeed, assuming henceforth the forward causal case of $\varrho \in]0, \infty[$, we have (see Theorem 1.1.4)[1]

$$\left\|\partial_0^{-1}\right\| \leq \frac{1}{\varrho}.$$

Remark 1.2.1 We note that the norm in $H_{\varrho,0}(\mathbb{R})$ is a Hilbert space variant of the Morgenstern norm, [36]. Based on the knowledge of the fundamental solution $h = \chi_{[0,\infty[}$ associated with ∂_0 we have on $L^\infty_{\text{loc}}(\mathbb{R})$-functions f with

$$\sup\left\{\exp\left(-\varrho t\right)|f\left(t\right)|\,;\ t \in \mathbb{R}\right\} < \infty,$$

that is, on $L^\infty_\varrho(\mathbb{R}) := \{f \in L^\infty_{\text{loc}}(\mathbb{R});\ \sup\{\exp(-\varrho t)|f(t)|\,;\ t \in \mathbb{R}\} < \infty\}$, that

$$\partial_0^{-1} = \chi_{[0,\infty[} * .$$

We recall that by Theorem 1.1.4 the same formula is true in $H_{\varrho,0}(\mathbb{R})$. The continuity on $L^\infty_\varrho(\mathbb{R})$ can be confirmed easily by estimating

$$\begin{aligned}\left|\exp\left(-\varrho t\right)\int_{-\infty}^{t} f\left(s\right)\,ds\right| &\leq \left|\exp\left(-\varrho t\right)\int_{-\infty}^{t} \exp\left(\varrho s\right)\,\exp\left(-\varrho s\right)|f\left(s\right)|\,ds\right|,\\ &\leq \left|\exp\left(-\varrho t\right)\int_{-\infty}^{t} \exp\left(\varrho s\right)\,ds\right| |f|_{L^\infty_\varrho(\mathbb{R})} = \frac{1}{\varrho}|f|_{L^\infty_\varrho(\mathbb{R})}\end{aligned}$$

for all $t \in \mathbb{R}$ and $f \in L^\infty_\varrho(\mathbb{R})$ and recalling that

$$\begin{aligned}\left(\partial_0^{-1} f\right)(t) &= \int_{\mathbb{R}} \chi_{[0,\infty[}\left(t-s\right) f\left(s\right)\,ds\\ &= \int_{-\infty}^{t} f\left(s\right)\,ds.\end{aligned}$$

[1] Indeed, one can even confirm that $\left\|\partial_{0,\varrho}^{-1}\right\| = \frac{1}{\varrho}$.