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New Sinc Methods of Numerical Analysis

Festschrift in Honor of Frank Stenger's 80th Birthday

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ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-030-49715-6 ISBN 978-3-030-49716-3 (eBook)
https://doi.org/10.1007/978-3-030-49716-3

Mathematics Subject Classification: 65-XX, 34A08, 65R20, 33F05

This book is published under the imprint Birkhäuser, www.birkhauser-science.com, by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Frank Stenger 2018

This book is compiled to honor
Frank Stenger's 80th birthday
to collect new developments in the field of
Sinc methods.

Preface

Frank Stenger, through his papers published in the 1960s and 1970s, is considered to be the founder of modern Sinc theory. He was born in Magyarpolány, Hungary in July 1938. To celebrate the 80th anniversary of his birthday, an international symposium in recognition of Stenger's major contributions to mathematics took place at Rhodes, Greece. The symposium was held from 13 to 18 September 2018 and was attended by participants from countries all over the world. It was organized by Gerd Baumann of the Mathematics Department of German University in Cairo. The symposium was devoted to Sinc theory and its new developments in numeric computations. The impact and development of this theory, from the origin to the present day, was the subject of a series of general presentations by leading experts in the field. The colloquium concluded with a workshop covering recent research in this highly active area.

Frank's work with Sinc methods began when he substantially revised a paper written by John McNamee and Ian Whitney on Whittaker's Cardinal Function in the 1960s. The work took off, as Sinc methods turned out to be an excellent tool for making approximations. The beautiful coinage of the Sinc function in the original paper (most likely due to McNamee) was "...a function of royal blood, whose distinguished properties separate it from its bourgeois brethren." Since then Sinc methods developed in a way allowing to solve problems in a wide range of areas in mathematics, physics, electrical, and fluid dynamic problems and is the primary tool used in wavelet applications. For Frank, Sinc methods have always been the center of his work and he coined it once as "...it's been a very, very lucky area in which to work ..." and he continues to work in this area up to this day.

The organizers of the symposium decided not to publish proceedings of the meeting in the usual form. Instead, it was planned to prepare, in conjunction with the symposium, a volume containing a complete bibliography of Stenger's published work, and to present the various aspects of Sinc theory at a rather general level making it accessible to the nonspecialist.

The present volume is a collection of 15 chapters relating to the symposium. It contains, in somewhat extended form, the survey lectures on Sinc theory given by the speakers. The contributions are divided into three parts incorporating applications, new developments, and bibliographic work. To complement the range of topics, the editor invited a few participants and coworkers of Frank Stenger to provide a review or other contributions in an area related to their current work covering some important aspects of current interest. Thus, the first part of the volume contains contributions which are application oriented using Sinc methods. The second part includes contributions which open the horizon to new fields and new developments. The volume ends with a comprehensive bibliography of Frank Stenger's work. We hope that these articles, besides being a tribute to Frank Stenger, will be a useful resource for researchers, graduate students, and others looking for an overview and new developments in the field of Sinc methods.

The articles in this volume can be read essentially independently. The authors have included cross-references to other sources. In order to respect the style of the authors, the editor did not ask them to use a uniform standard for notations and conventions of terminology.

As regards the present volume, we are grateful to our authors for all the efforts they have put into the project, as well as to our referees for generously giving of their time. We thank Nelson Beebe who undertook the immense task of preparing the bibliography for Frank's work. We are much indebted to Thomas Hempfling from Springer Verlag for continuing support in a fruitful and rewarding partnership.

Ulm, Germany Gerd Baumann
November 2019

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Part I
Applications

The first part of the book contains eight contributions which are mainly applications of Sinc methods. The different chapters demonstrate the variety of fields in which Sinc methods can be applied to problems in mathematics, in physics, in engineering, and in sociology. This part serves to demonstrate the strength of Sinc methods utilizing the variety of the computations.

Chapter 1
Sinc-Gaussian Approach for Solving the Inverse Heat Conduction Problem

M. H. Annaby and R. M. Asharabi

Abstract We introduce a new numerical method based on the sinc-Gaussian operator for solving the inverse heat equation. We establish rigorous proofs of the error estimates for both truncation and aliasing errors. The effect of the amplitude error, which has not been considered before, is also investigated theoretically and numerically for the first time in inverse heat problems. The domain of solvability of the inverse heat problem is enlarged and numerical examples show the superiority of the technique over the classical sinc-method. The power of the method is exhibited through several examples.

Keywords Inverse heat equation · Sinc-Gaussian sampling · Gaussian convergence factor · Gaussian convergence factor · Amplitude and truncation errors.

1.1 Introduction

The direct heat conduction problem consists in finding the temperature $u(x,t)$ which satisfies

$$\begin{aligned} \partial_t u(x,t) &= \partial_{xx} u(x,t), \qquad x \in \mathscr{J},\ t > 0, \\ u(x,0) &= f(x), \quad x \in \mathscr{J}, \end{aligned} \tag{1.1}$$

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G. Baumann (ed.), *New Sinc Methods of Numerical Analysis*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-49716-3_1

where f is a given function. Here $\mathscr{J} = (0, \infty)$ or $\mathscr{J} = \mathbb{R}$. The problem may be analytically solved via (in the case $\mathscr{J} = \mathbb{R}$)

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t}\right) f(y)dy, \tag{1.2}$$

provided that f is well behaved. For instance (1.2) is well defined if $f \in L^2(\mathbb{R})$, cf. [11, 22].

The inverse heat problem, which we are considering in this paper, is to determine f from a known solution $u(x,t)$. The authors of [8] proposed a sinc-interpolation method to solve the inverse problem. Their procedure can be outlined as follows. For $d > 0$ and $\mathscr{S}_d \subset \mathbb{C}$ being the infinite strip $\mathscr{S}_d := \{z \in \mathbb{C} : |\Im z| < d\}$, we define $\mathscr{H}^p(\mathscr{S}_d),\ 1 \le p < \infty$, as the set of holomorphic functions on $\mathscr{S}_d$ such that if $D_d(\varepsilon)$ is defined for $0 < \varepsilon < 1$ by

$$D_d(\varepsilon) = \{z \in \mathbb{C} : |\Re z| < 1/\varepsilon, |\Im z| < d(1-\varepsilon)\},$$

then $N^p(f, \mathscr{S}_d) < \infty,\ 1 \le p < \infty$, where

$$N^p(f, \mathscr{S}_d) := \lim_{\varepsilon \to 0} \left(\int_{\partial D_d(\varepsilon)} |f(z)|^p |dz|\right)^{1/p}. \tag{1.3}$$

Gilliam et al. [8] proposed a solution to the inverse heat problem (1.1) under the restriction $f \in \mathscr{H}^1(\mathscr{S}_d)$. This restriction via relaxed below. The solution of [8] is based on the expandability of f via the sinc-interpolation series

$$f(y) \simeq \sum_{n=-\infty}^{\infty} f(nh)\, \mathrm{sinc}\left(\frac{y-nh}{h}\right), \tag{1.4}$$

where $h > 0$ is a fixed step-size and the sinc function is defined by

$$\mathrm{sinc}(t) := \begin{cases} \dfrac{\sin \pi t}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{1.5}$$

If we define the aliasing error $\mathscr{E}[f](y)$ via

$$\mathscr{E}[f](y) = f(y) - \sum_{n=-\infty}^{\infty} f(nh)\mathrm{sinc}\left(\frac{y-nh}{h}\right), \tag{1.6}$$

then [18, Theorem 3.1.3] for $f \in \mathscr{H}^1(\mathscr{S}_d)$, $\mathscr{E}[f](y)$ is bounded via

$$\|\mathscr{E}[f]\|_\infty \le \frac{N^1(f, \mathscr{S}_d)}{2\pi d \sinh(\pi d/h)} = O\left(\exp\left(-\frac{\pi d}{h}\right)\right), \quad \text{as } h \to 0^+, \tag{1.7}$$

where the infinity norm is defined by $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. Moreover if the function f decays as

$$|f(x)| \le c \exp(-a|x|), \quad x \in \mathbb{R}, \tag{1.8}$$

for some positive constants c and a, and one selects $h = \sqrt{\frac{\pi d}{aN}}$, then the combined aliasing and truncation error which is given by

$$\mathscr{E}_N[f](y) = f(y) - \sum_{|n|\le N} f(nh)\text{sinc}\left(\frac{y - nh}{h}\right), \tag{1.9}$$

for some $N \in \mathbb{N}$ is estimated in [18, Theorem 3.1.7] to be

$$\|\mathscr{E}_N[f]\|_\infty \le C\sqrt{N} \exp\left(-\sqrt{\pi a d N}\right), \quad \text{as } N \to \infty, \tag{1.10}$$

where the constant C depends only on f, a and d.

The technique of [8], see also [10, pp. 27–31], stands on approximating $\{f(nh)\}$ for $-N \le n \le N$ through solving the following truncated linear system of equations

$$B_N\mathbf{f} = 2\pi\mathbf{u} \iff \sum_{n=-N}^{N} f(nh)\beta_{n-k} = 2\pi u(kh, \widetilde{t}), \quad -N \le k \le N, \tag{1.11}$$

where $B_N = \left(b_{jk}\right)_{-N\le j,k\le N} = \left(\beta_{j-k}\right)_{-N\le j,k\le N}$,

$$\mathbf{f} = (f(-Nh), f((-N+1)h), \ldots, f(0), \ldots, f((N-1)h), f(Nh))^\top, \tag{1.12}$$

$$\mathbf{u} = \left(u(-Nh, \widetilde{t}), u((-N+1)h, \widetilde{t}), \ldots, u(0, \widetilde{t}) \ldots, u((N-1)h, \widetilde{t}), u(Nh, \widetilde{t})\right)^\top, \tag{1.13}$$

$$\beta_l := \int_{-\pi}^{\pi} \exp(il\tau) \exp\left(-\frac{\tau^2}{4\pi^2}\right) d\tau, \quad -2N \le l \le 2N, \tag{1.14}$$

and $\widetilde{t} := \left(\frac{h}{2\pi}\right)^2$. Hereafter, $A^\top$ denotes the transpose of a matrix A.

While the paper [8] nicely connects the sinc-method to the inverse heat problem, the authors did not investigate the amplitude error resulting from the effect of approximating the samples $\{f(nh)\}_{n=-N}^{N}$ by $\left\{\widetilde{f}(nh)\right\}_{n=-N}^{N}$. Moreover, in the analysis of the stability of (1.11), it is proved that $|\beta_l| \le \frac{1}{l}$, $l \ge 1$. However, this estimate does not imply the boundedness of the operator $B = \lim_{N\to\infty} B_N$ on ℓ^2. Thus neither the existence of B^{-1} nor the stability of the system is not theoretically guaranteed. However, in [8] the authors introduced satisfactory computations of numerical condition numbers. On the other hand, the connection between sinc-interpolations and inverse problems is rarely investigated, cf. e.g. [16, 23]. As far as we know, no studies have been performed about the connection between the sinc-Gaussian method and the inverse heat conduction. Therefore one may ask about the possibility of applying recent advances in the sinc-methods for the inverse heat problem to improve error analysis as well as convergence rates.

We aim in this paper to use the sinc-Gaussian operator, defined by Wei et al. [20, 21], and developed by Qian et al. [12–14], Schmeisser and Stenger [15], Tanaka et al. [19], and the authors [2, 4, 5], to solve the inverse heat problem. As expected this technique will have the following advantages:

- No need for infinite systems.
- Acceleration the rate of convergence to $O\left(\exp\left(-\frac{\pi N}{2}\right)/\sqrt{N}\right)$ which is independent of d or h as the bound (1.7) of sinc-interpolation.
- The localization property that allows to approximate f in appropriately desired domains.

The other task of this paper is to investigate the effect of the amplitude error due to use approximating samples on the solution of the inverse problem. We investigate the amplitude error for the method of [8] and for the sinc-Gaussian interpolation established here. Numerical examples and illustrations are introduced in the last section.

1.2 Sinc-Type Errors

In this section we demonstrate the types of error associated with the use of the sinc-method in the approximation of the inverse heat problem. Let

$$\widetilde{\mathscr{E}}[f](y) := f(y) - \sum_{n=-\infty}^{\infty} \widetilde{f}(nh)\operatorname{sinc}\left(\frac{y-nh}{h}\right), \tag{1.15}$$

where $\widetilde{f}(nh)$ are approximations of the samples $f(nh),\ n \in \mathbb{Z}$, such that there is a sufficiently small ε, which satisfies $\varepsilon_n := |f(nh) - \widetilde{f}(nh)| < \varepsilon$ for all $n \in \mathbb{Z}$ and

$$\varepsilon_n \le |f(nh)|, \quad n \in \mathbb{Z}. \tag{1.16}$$

In the following we assume that the decay condition (1.16) is fulfilled and $\varepsilon_n < \varepsilon$, for all $n \in \mathbb{Z}$.

Theorem 1.1 *Let $f \in \mathscr{H}^1(\mathscr{S}_d)$ satisfy the decay condition*

$$|f(t)| \le \frac{C_f}{|t|^{\alpha+1}}, \quad \alpha \in]0,1], \quad |t| > 1, \tag{1.17}$$

where C_f is a positive constant. Then we have for $0 < \varepsilon < \min\{h, h^{-1}, 1/\sqrt{e}\}$ and $y \in \mathbb{R}$,

$$\left|\mathscr{E}[f](y) - \widetilde{\mathscr{E}}[f](y)\right| \le \frac{4}{\alpha+1}\left(3^{(\alpha+1)/2}\, e + C_f 2^{(\alpha+1)/2} e^{1/4}\right) \varepsilon \log(1/\varepsilon), \tag{1.18}$$

where $\alpha \in]0,1[$ and the constant C_f depends only on f.

Proof Let $p, q > 1,\ 1/p + 1/q = 1$. For $y \in \mathbb{R}$, we apply Hölder's inequality, then we use an inequality of Splettstößer et al. [17],

$$\left(\sum_{n=-\infty}^{\infty} \left|\operatorname{sinc}\left(\frac{y-nh}{h}\right)\right|^q\right)^{1/q} < p, \tag{1.19}$$

to obtain

$$\left|\mathscr{E}[f](y) - \widetilde{\mathscr{E}}[f](y)\right| \le p\left(\sum_{n=-\infty}^{\infty} |\varepsilon_n|^p\right)^{1/p}. \tag{1.20}$$

Using the conditions (1.16) and (1.17) for a sufficiently small ε, the technique of [6], see also [1, 3], we obtain the estimate

$$p\left(\sum_{n=-\infty}^{\infty} |\varepsilon_n|^p\right)^{1/p} \le \frac{4}{\alpha+1}\left(3^{(\alpha+1)/2}\, e + C_f 2^{(\alpha+1)/2} e^{1/4}\right) \varepsilon \log(1/\varepsilon). \tag{1.21}$$

Combining (1.21) and (1.20) immediately implies (1.18).

For convenience, we define $A_\varepsilon[f]$ to be

$$A_\varepsilon[f] := \frac{4}{\alpha+1}\left(3^{(\alpha+1)/2}\, e + C_f 2^{(\alpha+1)/2} e^{1/4}\right) \varepsilon \log(1/\varepsilon). \tag{1.22}$$

After we derived an estimate for the amplitude error (1.18), we can now estimate the error that arises from applying the sinc-method in the inverse problem. We consider both aliasing and amplitude errors.

Corollary 1.1 *Let $f \in \mathscr{H}^1(\mathscr{S}_d)$. Then we have for $0 < \varepsilon < \min\{h, h^{-1}, 1/\sqrt{e}\}$*

$$\|\widetilde{\mathscr{E}}[f]\|_\infty \leq \frac{N^1(f, \mathscr{S}_d)}{2\pi d \sinh(\pi d/h)} + A_\varepsilon[f], \tag{1.23}$$

where $N^1(f, \mathscr{S}_d)$ is defined in (1.3).

Proof The estimate (1.23) directly follows from combining the estimates (1.18) and (1.7).

To apply the sinc-method in the inverse heat problem, we have only a finite number of observations. Therefore a truncation error arises. In the following we estimate $\widetilde{\mathscr{E}}_N[f](y)$ for a positive integer N, where

$$\widetilde{\mathscr{E}}_N[f](y) := f(y) - \sum_{|n|\leq N} \widetilde{f}(nh)\text{sinc}\left(\frac{y - nh}{h}\right). \tag{1.24}$$

Corollary 1.2 *Let $f \in \mathscr{H}^1(\mathscr{S}_d)$ obey the decay (1.8). Then for $0 < \varepsilon < \min\{h, h^{-1}, 1/\sqrt{e}\}$ we have*

$$\|\widetilde{\mathscr{E}}_N[f]\|_\infty \leq C\sqrt{N}\exp\left(-\sqrt{\pi a d N}\right) + A_\varepsilon[f], \tag{1.25}$$

where C is a positive constant that depends only on f, α and d.

Proof From the triangle inequality, we obtain

$$\|\widetilde{\mathscr{E}}_N[f]\|_\infty \leq \|\mathscr{E}_N[f]\|_\infty + \|\mathscr{E}[f] - \widetilde{\mathscr{E}}[f]\|_\infty. \tag{1.26}$$

Combining (1.18), (1.10) and (1.26) implies (1.25).

A more general case, where the decay condition (1.8) by the relaxed one (1.17) is treated in the following theorem.

Theorem 1.2 *Let $f \in \mathscr{H}^1(\mathscr{S}_d)$ such that (1.17) is fulfilled. Then we have for $h = \sqrt{\frac{\pi d}{N}}$*

$$\|\mathscr{E}_N[f]\|_\infty \leq N^1(f, \mathscr{S}_d)\, e^{-\sqrt{\pi d N}} + \frac{2}{\alpha(\pi d N)^{\alpha/2}}. \tag{1.27}$$

Proof From the definition of $\mathscr{E}_N[f]$, we obtain

$$\|\mathscr{E}_N[f]\|_\infty \leq \|\mathscr{E}[f]\|_\infty + \left\|\sum_{|n|>N} f(nh)sinc\left(\frac{\cdot - nh}{h}\right)\right\|_\infty. \tag{1.28}$$

The first term on the right-hand side of (1.28) has been estimated in (1.7). Letting $h = \sqrt{\frac{\pi d}{N}}$ in (1.7) implies

$$\|\mathscr{E}[f]\|_\infty \le N^1(f, \mathscr{S}_d)\, e^{-\sqrt{\pi d N}}. \tag{1.29}$$

We now estimate the second term on the right-hand side of (1.28). Since f obeys the decay condition (1.17), we obtain

$$\left|\sum_{|n|>N} f(nh)\mathrm{sinc}\left(\frac{y-nh}{h}\right)\right| \le \sum_{|n|>N} |f(nh)| \le 2\int_{Nh}^{\infty} \frac{1}{t^{\alpha+1}} dt = \frac{2}{\alpha(\pi d N)^{\alpha/2}}, \tag{1.30}$$

where we have used in the last step that $h = \sqrt{\frac{\pi d}{N}}$. Combining (1.30) and (1.29) implies (1.27).

Considering the amplitude error leads to the following corollary.

Corollary 1.3 *Let $f \in \mathscr{H}^1(\mathscr{S}_d)$ for which the decay condition (1.17) is satisfied. Then we have for $h = \sqrt{\frac{\pi d}{N}}$,*

$$\|\widetilde{\mathscr{E}}_N[f]\|_\infty \le N^1(f, \mathscr{S}_d)\, e^{-\sqrt{\pi d N}} + \frac{2}{\alpha(\pi d N)^{\alpha/2}} + A_\varepsilon[f]. \tag{1.31}$$

Proof Results directly from combining (1.27), (1.26) and (1.18).

The following theorem is estimating the amount of a function in the amplitude error, which can be made as small as wished. It is derived under the assumption that B_N is invertible.

Theorem 1.3 *Let B_N be invertible and $\widetilde{B}_N$ be a perturbed matrix for which*

$$\|B_N - \widetilde{B}_N\| < \delta \ll \frac{1}{\|B_N^{-1}\|}, \tag{1.32}$$

where $\|.\|$ is the Euclidean norm. Then $\widetilde{B}_N$ is also invertible and if

$$B_N \mathbf{f} = 2\pi \mathbf{u}, \quad \widetilde{B}_N \widetilde{\mathbf{f}} = 2\pi \mathbf{u},$$

then

$$\|\mathbf{f} - \widetilde{\mathbf{f}}\| \le \frac{2\pi\delta \|B_N^{-1}\|\|\mathbf{u}\|}{1 - \|B_N^{-1}\|\delta}, \tag{1.33}$$

which goes to zero as $\delta \to 0$.

Proof Results directly from [9, p. 71].

1.3 Sinc-Gaussian Heat Inversion on $\mathbb{R}$

Let $\mathfrak{B}(\mathscr{S}_d)$ be the class of holomorphic functions on $\mathscr{S}_d$ which are bounded on $\mathbb{R}$. Let $\mathscr{E}_2$ be the class of all entire functions that belong to $L^2(\mathbb{R})$ when restricted to the real axis. On the class $\mathfrak{B}(\mathscr{S}_d)$, Schmeisser and Stenger defined in [15] a sinc-Gaussian sampling operator $\mathscr{G}_{\mathrm{h},N} : \mathfrak{B}(\mathscr{S}_d) \to \mathscr{E}_2$ by

$$\mathscr{G}_{\mathrm{h},N}[f](z) := \sum_{n\in\mathbb{Z}_N(z)} f(n\mathrm{h})\,\mathrm{sinc}\left(\frac{z-n\mathrm{h}}{\mathrm{h}}\right)\exp\left(-\frac{\pi}{2N}\left(\frac{z-n\mathrm{h}}{\mathrm{h}}\right)^2\right), \quad z\in\mathbb{C}, \tag{1.34}$$

where N is a positive integer, $\mathbb{Z}_N(z) := \Big\{n \in \mathbb{Z} : |\lfloor \mathrm{h}^{-1}\Re z + 1/2\rfloor - n| \le N\Big\}$, $\lfloor\cdot\rfloor$ denotes the floor function and $\mathrm{h} = \frac{d}{N}$. The authors of [15] have bounded the truncation error $\big|f(z) - \mathscr{G}_{\mathrm{h},N}[f](z)\big|$ when $f \in \mathfrak{B}(\mathscr{S}_d)$ and $z \in \mathscr{S}_{d/4}$. Here we state only the real version of this bound because our technique will be entirely on $\mathbb{R}$. If $f \in \mathfrak{B}(\mathscr{S}_d)$, then we have, cf.[15, Theorem 3.1],

$$\big\|f - \mathscr{G}_{\mathrm{h},N}[f]\big\|_\infty \le 4\sqrt{2}\|f\|_\infty \frac{\mathrm{e}^{-\frac{\pi}{2}N}}{\pi\sqrt{N}}. \tag{1.35}$$

Since the samples $\{f(n\mathrm{h})\}_{n\in\mathbb{Z}_N(z)}$ cannot be measured explicitly for most applied problems, and alternative approximate ones $\big\{\widetilde{f}(n\mathrm{h})\big\}_{n\in\mathbb{Z}_N(z)}$ are measured, an amplitude error appears. Let

$$\mathscr{G}_{\mathrm{h},N}[\widetilde{f}](x) := \sum_{n\in\mathbb{Z}_N(x)} \widetilde{f}(n\mathrm{h})\,\mathrm{sinc}\left(\frac{x-n\mathrm{h}}{\mathrm{h}}\right)\exp\left(-\frac{\pi}{2N}\left(\frac{x-n\mathrm{h}}{\mathrm{h}}\right)^2\right), \quad x\in\mathbb{R}. \tag{1.36}$$

The authors of [2] established a bound of the amplitude error as follows:

$$\big\|\mathscr{G}_{\mathrm{h},N}[f] - \mathscr{G}_{\mathrm{h},N}[\widetilde{f}]\big\|_\infty \le 2\,\varepsilon\ e^{-\pi/8N}\left(1+\sqrt{2N/\pi^2}\right), \tag{1.37}$$

where ε is sufficiently small that satisfies $|f(n\mathrm{h}) - \widetilde{f}(n\mathrm{h})| < \varepsilon$ for all $n \in \mathbb{Z}_N(z)$.

In the following, we introduce a new technique based on sin-Gaussian interpolation to solve the inverse heat problem. Let the initial function f of (1.1) belongs to $\mathfrak{B}(\mathscr{S}_d)$. Then $f \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ when the domain of f is restricted on the real line. Therefore the solution (1.2) is well defined and $u \in C^\infty(\mathbb{R}\times(0,\infty))$, cf. [7, p. 47]. We assume that the solution of the inverse heat problem is based on approximating f via the sinc-Gaussian interpolation (1.34). Therefore $f(y) \simeq \mathscr{G}_{\mathrm{h},N}[f](y)$, and consequently as in [8, 10]

$$u(x,t) \simeq \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}_N(y)} f(n\mathrm{h}) \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t}\right) \operatorname{sinc}\left(\frac{y}{\mathrm{h}} - n\right) e^{-\frac{\pi}{2N}\left(\mathrm{h}^{-1}y-n\right)^2} dy. \tag{1.38}$$

Letting $x = k\mathrm{h}$, $s = \frac{y-k\mathrm{h}}{\mathrm{h}}$ and $l = n - k$ in (1.38) yields

$$u(k\mathrm{h},t) \simeq \frac{\mathrm{h}}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}_N(y)} f(n\mathrm{h}) \int_{-\infty}^{\infty} e^{-\frac{(\mathrm{h}s)^2}{4t}} \operatorname{sinc}(s-l)\, e^{-\frac{\pi(s-l)^2}{2N}} ds. \tag{1.39}$$

The Sinc function is merely the Fourier coefficient

$$\operatorname{sinc}(s-l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-is\tau} e^{il\tau} d\tau. \tag{1.40}$$

Combining (1.40) and (1.39) implies

$$u(k\mathrm{h},t) \simeq \frac{\mathrm{h}}{2\pi\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}_N(y)} f(nh) \int_{-\pi}^{\pi} e^{il\tau} \int_{-\infty}^{\infty} e^{-\left(is\tau + \frac{\mathrm{h}^2 s^2}{4t} + \frac{\pi(s-l)^2}{2N}\right)} ds d\tau, \tag{1.41}$$

where $l = n - k$. Calculating the infinite integral in (1.41) and letting $t_0 := \left(\frac{d}{2\pi N}\right)^2$, we obtain the system of equations

$$u(k\mathrm{h},t_0) = \frac{1}{2\sqrt{\pi^2 + \pi/2N}} \sum_{n\in\mathbb{Z}_N(y)} f(n\mathrm{h})\mathscr{B}_{n-k}, \quad k \in \mathbb{Z}_N(y), \tag{1.42}$$

where

$$\mathscr{B}_l = e^{\frac{-l^2\pi^2}{2\pi N+1}} \int_{-\pi}^{\pi} e^{il\tau} \exp\left(-\frac{2i\pi l\tau + L\tau^2}{2\pi(2\pi N+1)}\right) d\tau. \tag{1.43}$$

The system (1.42) can be written in a more compact form as

$$\mathbf{B}_N \mathbf{f} = a_N \mathbf{u}, \tag{1.44}$$

where $a_N := 2\sqrt{\pi^2 + \pi/2N}$ and $\mathbf{B}_N$ is the $(2N+1)\times(2N+1)$ symmetric Toeplitz matrix

$$\mathbf{B}_N = \begin{pmatrix} \mathscr{B}_0 & \mathscr{B}_1 & \mathscr{B}_2 & \dots & \mathscr{B}_{2N} \\ \mathscr{B}_1 & \mathscr{B}_0 & \mathscr{B}_1 & \dots & \mathscr{B}_{2N-1} \\ \mathscr{B}_2 & \mathscr{B}_1 & \mathscr{B}_0 & \dots & \mathscr{B}_{2N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathscr{B}_{2N} & \mathscr{B}_{2N-1} & \mathscr{B}_{2N-2} & \dots & \mathscr{B}_0 \end{pmatrix}. \tag{1.45}$$

The symmetry of the Toeplitz matrix comes from the property $\mathscr{B}_{-l} = \mathscr{B}_l$, for all $0 \le l \le 2N$. With $N_y := \lfloor h^{-1}y + 1/2 \rfloor$, $(2N+1)$-vectors $\mathbf{f}$ and $\mathbf{u}$ are given by

$$\mathbf{f} = \big(f((-N+N_y)\mathrm{h}), \dots, f((N+N_y)\mathrm{h})\big)^\top, \tag{1.46}$$

$$\mathbf{u} = \big(u((-N+N_y)\mathrm{h}, t_0), \dots, u((N+N_y)\mathrm{h}, t_0)\big)^\top. \tag{1.47}$$

Assume that $\mathbf{u}$ is known and that we determine $\mathbf{f}$ from (1.44). We compute the integrals $\mathscr{B}_l$, $-2N \le l \le 2N$, which are the elements of the matrix $\mathbf{B}_N$, numerically because they can not be computed exactly. Again the amplitude error appears. In this setting, we do not need to consider the case of infinite series (1.4). However, as in the previous section, we assume the invertibility of $\mathbf{B}_N$. We then have the following theorem.

Theorem 1.4 *Let $f \in \mathfrak{B}(\mathscr{S}_d)$, let $\widetilde{\mathbf{B}}_N$ be a perturbed matrix from $\mathbf{B}_N$ such that*

$$\|\mathbf{B}_N - \widetilde{\mathbf{B}}_N\| < \delta \ll \frac{1}{\|\mathbf{B}_N^{-1}\|}, \tag{1.48}$$

where $\|\cdot\|$ is the matrix norm. Then

$$\|f - \mathscr{G}_{\mathrm{h},N}[\widetilde{f}]\|_\infty \le 4\sqrt{2}\|f\|_\infty \frac{\mathrm{e}^{-\frac{\pi}{2}N}}{\pi\sqrt{N}} + 2\delta \frac{a_N \|\mathbf{B}_N^{-1}\|\|\mathbf{u}\|}{1 - \|\mathbf{B}_N^{-1}\|\delta}\ e^{-\pi/8N}\left(1 + \sqrt{2N/\pi^2}\right). \tag{1.49}$$

Proof Since $\mathbf{B}_N$ is invertible and the condition (1.48) is satisfied,

$$\|\mathbf{f} - \widetilde{\mathbf{f}}\| \le \frac{\delta a_N \|\mathbf{B}_N^{-1}\|\|\mathbf{u}\|}{1 - \|\mathbf{B}_N^{-1}\|\delta}. \tag{1.50}$$

Hence, applying the triangle inequality, we obtain

$$\|f - \mathscr{G}_{\mathrm{h},N}[\widetilde{f}]\|_\infty \le \|f - \mathscr{G}_{\mathrm{h},N}[f]\|_\infty + \|\mathscr{G}_{\mathrm{h},N}[f] - \mathscr{G}_{\mathrm{h},N}[\widetilde{f}]\|_\infty. \tag{1.51}$$

Bounds on the first and second terms of (1.51) are given in (1.35) and (1.37), respectively. Combining (1.35), (1.37) and (1.51) with $\varepsilon := \frac{\delta a_N \|\mathbf{B}_N^{-1}\| \|\mathbf{u}\|}{1-\|\mathbf{B}_N^{-1}\|\delta}$ leads to (1.49).

1.4 Sinc-Gaussian Heat Inversion on $\mathbb{R}^+$

In this section we treat the problem (1.1) with $\mathscr{J} = (0, \infty)$. The solution of (1.1) is given by, cf. e.g. [8],

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left(\exp\left(\frac{-(x-y)^2}{4t} \right) - \exp\left(\frac{-(x+y)^2}{4t} \right) \right) f(y) dy. \tag{1.52}$$

Let F be the odd extension of f on $\mathbb{R}$

$$F(y) = \begin{cases} f(y), & \text{if } y \geq 0, \\ -f(-y), & \text{if } y < 0. \end{cases} \tag{1.53}$$

For a continuous initial data f, it is necessary that $f(0) = 0$. The solution (1.52) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty \exp\left(\frac{-(x-y)^2}{4t} \right) F(y) dy. \tag{1.54}$$

If F belongs to the class $C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the solution in (1.54) is well defined and $F(y) \simeq \mathscr{G}_{\mathrm{h},N}[F](y)$, i.e.

$$F(y) \simeq \sum_{n=N_y-N}^{N_y+N} F(n\mathrm{h}) \operatorname{sinc}\left(\frac{y-n\mathrm{h}}{\mathrm{h}} \right) \mathrm{e}^{-\frac{\pi}{2N}\left(\frac{y-n\mathrm{h}}{\mathrm{h}}\right)^2}. \tag{1.55}$$

Substituting from (1.55) into (1.54) and using technique as in the former, we obtain the following system of equations

$$\sum_{n=N_y-N}^{N_y+N} F(n\mathrm{h}) \mathscr{B}_{n-k} = a_N u(k\mathrm{h}, t_1), \qquad N_y - N \leq k \leq N_y + N, \tag{1.56}$$

where $t_1 := \left(\frac{d}{2\pi N}\right)^2$, $a_N := 2\sqrt{\pi^2 + \pi/2N}$ and $\mathscr{B}_l$ is defined in (1.43). In the case $N_y > N$, the $(2N+1) \times (2N+1)$ system of equations (1.56) becomes (1.44) because $F(y) = f(y)$ on $[0, \infty)$. Therefore, we solve this system in the same way as in the former to find the vector $\mathbf{f}$. If $N_y = N$, the system (1.56) reduces to $2N \times 2N$ of equations because $f(0) = 0$ and $u(0, t_1) = 0$ which comes from the odd extension of f. When $0 \le N_y < N$, the order of a matrix in the system (1.56) reduces to $N + N_y$. Recall that $\mathscr{B}_{-l} = \mathscr{B}_l$ for all $-N \le l \le N$, $F(-nh) = -f(nh)$ and $u(-nh, t_1) = -u(nh, t_1)$ for all $-(N - N_y) \le n \le N - N_y$. Hence the system (1.56) has the block form

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \mathbf{A}_2^\top & \mathbf{A}_1 & \mathbf{A}_4 \\ \mathbf{A}_3^\top & \mathbf{A}_4^\top & \mathbf{A}_5 \end{pmatrix} \begin{pmatrix} -\mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = a_N \begin{pmatrix} -\mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix}, \tag{1.57}$$

where the matrices $\mathbf{A}_j,\ j = 1, \ldots, 5$ are defined by

$$\mathbf{A}_1 := \begin{pmatrix} \mathscr{B}_0 & \ldots & \mathscr{B}_{N-N_y-1} \\ \vdots & & \vdots \\ \mathscr{B}_{N-N_y-1} & \ldots & \mathscr{B}_0 \end{pmatrix}, \qquad \mathbf{A}_2 := \begin{pmatrix} \mathscr{B}_{N-N_y+1} & \ldots & \mathscr{B}_{2(N-N_y)} \\ \vdots & & \vdots \\ \mathscr{B}_2 & \ldots & \mathscr{B}_{N-N_y+1} \end{pmatrix},$$

$$\mathbf{A}_3 := \begin{pmatrix} \mathscr{B}_{2(N-N_y)+1} & \ldots & \mathscr{B}_{2N} \\ \vdots & & \vdots \\ \mathscr{B}_{N-N_y+2} & \ldots & \mathscr{B}_{N+N_y+1} \end{pmatrix}, \qquad \mathbf{A}_4 := \begin{pmatrix} \mathscr{B}_{N-N_y} & \ldots & \mathscr{B}_{N+N_y-1)} \\ \vdots & & \vdots \\ \mathscr{B}_1 & \ldots & \mathscr{B}_{2N_y} \end{pmatrix},$$

$$\mathbf{A}_5 := \begin{pmatrix} \mathscr{B}_0 & \ldots & \mathscr{B}_{2N_y-1} \\ \vdots & & \vdots \\ \mathscr{B}_{2N_y-1} & \ldots & \mathscr{B}_0 \end{pmatrix}.$$

The matrices $\mathbf{A}_j,\ j = 1, 2$ are of order $N - N_y$ while $\mathbf{A}_j,\ j = 3, 4$ are of order $(N - N_y) \times 2N_y$ and $\mathbf{A}_5$ has order $2N_y$. The vectors $\mathbf{f}_j$ and $\mathbf{u}_j,\ j = 1, 2, 3$ are defined by

$$\mathbf{f}_1 = \left(f\left(N - N_y)\mathrm{h}\right), \ldots, f(\mathrm{h})\right)^\top,$$
$$\mathbf{f}_2 = \left(f(\mathrm{h}), \ldots, f\left(N - N_y)\mathrm{h}\right)\right)^\top,$$
$$\mathbf{f}_3 = \left(f\left(N - N_y + 1)\mathrm{h}\right), \ldots, f\left(N + N_y)\mathrm{h}\right)\right)^\top,$$

and

$$\mathbf{u}_1 = \left(u\left(N - N_y)\mathrm{h}, t_0\right), \ldots, u(\mathrm{h}, t_0)\right)^\top,$$
$$\mathbf{u}_2 = \left(u(\mathrm{h}, t_0), \ldots, u\left(N - N_y)\mathrm{h}, t_0\right)\right)^\top,$$
$$\mathbf{u}_3 = \left(u\left(N - N_y + 1)\mathrm{h}, t_0\right), \ldots, u\left(N + N_y)\mathrm{h}, t_0\right)\right)^\top.$$

The vectors $\mathbf{f}_j,\ \mathbf{u}_j,\ j = 1, 2$ are of dimension $N - N_y$ while $\mathbf{f}_3$ and $\mathbf{u}_3$ are of dimension $2N_y$. Thus the matrices $\mathbf{A}_j,\ j = 3, 4, 5$ and the vectors $\mathbf{f}_3$ and $\mathbf{u}_3$ disappear from the system (1.57) when $N_y = 0$. Likewise, $\mathbf{A}_j,\ j = 1, 2,\ \mathbf{f}_1$ and $\mathbf{u}_1$ disappear from the system when $N_y = N$. Now, it is easy to see that

$$\mathbf{u}_2 = J_{N-N_y}\mathbf{u}_1, \quad \mathbf{f}_2 = J_{N-N_y}\mathbf{f}_1, \quad J_{N-N_y}\mathbf{A}_2^\top = \mathbf{A}_2 J_{N-N_y}, \quad J_{N-N_y}\mathbf{A}_1 J_{N-N_y} = \mathbf{A}_1, \tag{1.58}$$

where J_{N-N_y} is the $(N - N_y) \times (N - N_y)$ matrix defined as

$$J_{N-N_y} := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Using the relations (1.58), the system (1.57) reduces to

$$\begin{aligned} &2\left(\mathbf{A}_1 - \mathbf{A}_2 J_{N-N_y}\right)\mathbf{f}_1 - \left(\mathbf{A}_3 - J_{N-N_y}\mathbf{A}_4^\top\right)\mathbf{f}_3 = 2a_N\mathbf{u}_1, \\ &-\left(\mathbf{A}_3^\top - \mathbf{A}_4^\top J_{N-N_y}\right)\mathbf{f}_1 + \mathbf{A}_5\mathbf{f}_3 = a_N\mathbf{u}_3, \end{aligned} \tag{1.59}$$

which is of order $(N + N_N) \times (N + N_N)$. In the special case $N_y = 0$, the system (1.59) will be $(\mathbf{A}_1 - \mathbf{A}_2 J_N)\,\mathbf{f}_1 = a_N\mathbf{u}_1$.

1.5 Numerical Examples

We work out four numerical examples in this section. Examples 1.1–1.3 are considered in [8, 10]. The fourth example is devoted to an inverse heat problem on $(0, \infty)$. In all examples, we compare between the results of both sinc and sinc-Gaussian methods. Let $\widetilde{S}_N(x)$ to be

$$\widetilde{S}_N(x) := \sum_{|n|\le N} \widetilde{f}(nh)\operatorname{sinc}\left(\frac{y - nh}{h}\right),$$

and $x_k := d(k - 1/2)/N$. The bound of the classical technique is calculated using (1.7) with $h = \sqrt{\frac{\pi d}{N}}$. In all examples, with both techniques we choose $d = 1$. The condition numbers of the matrix $\mathbf{B}_N$ are given below in Table 1.1. They are very close to those computed for B_N in [8]. This indicates that both systems, (1.44) and (1.11) have similar stability properties.