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To my wife, Nazanin, and my son, Alireza, whose valuable support and patience have enabled me to complete this project, and to my parents for their encouragement, and finally to my mentor Prof. Hashem Rafii-Tabar, who gave me the opportunity to learn from him and work alongside him.

Esmaeal Ghavanloo

To my wife, Zahra, and my children, Ali and Yeganeh, for their patience and unwavering support.

S. Ahmad Fazelzadeh

To Angela for her unwavering support, patience, and understanding

Francesco Marotti de Sciarra
Foreword

The present book “Size-Dependent Continuum Mechanics Approaches: Theory and Application” is a collection of papers edited by Esmaeal Ghavanloo, S. Ahmad Fazelzadeh and Francesco Marotti de Sciarra. It covers a lot of interesting information about continuum mechanics approaches taking into account size-dependent effects. Approaches present in the book are mostly based on nonlocal theories, but also on micromorphic, peridynamic and gradient theories. Instead of the classical continuum mechanics with taking into account the material point and its infinitesimal surrounding, the alternative theories include also far-distance (beyond the infinitesimal surrounding) information. The majority of presentations in this book are related to contributions in this field of Ahmed Cemal Eringen (see, for example, W.H. Müller, “Eringen, Ahmed Cemal”, In H. Altenbach, A. Öchsner (eds), Encyclopedia of Continuum Mechanics, Springer, 2020, 860–862).

It should be noted that the non-classical continuum mechanics approaches are much more complicated in comparison with the classical ones. Many examples are presented in the book for beams and one can follow the discussion in a simple manner having only basic knowledge on the Euler-Bernoulli beam theory. In addition, several applications are given and one can see that non-classical approaches are helpful if the structural size is very small. In this case, size effects are obvious and classical theories failed. The advanced theories allow the solution of new problems, but they have also disadvantages: the number of constitutive parameters increases. The estimation of the constitutive parameters is not trivial and several suggestions are discussed in the literature.

The book contains 15 papers prepared by leading scientists in size-dependent theories. In the first chapter “Lattice-Based Nonlocal Elastic Structural Models” in the sense of Lagrange, Hencky and Eringen are discussed and compared. In the following chapter, “Eringen’s Nonlocal Integral Elasticity and Applications for Structural Models” are presented. The focus of applications is on carbon nanotubes. The third chapter is devoted to “Nonlocal Mechanics in the Framework of the General Nonlocal Theory”. Here the focus is on the general nonlocal theory and it is shown that the strain gradient theory can capture the same phenomena. In the next
chapter “Displacement Based Nonlocal Models for Size Effect Simulation in Nanomechanics” are presented. The main results of such theory and the differences with other nonlocal models are described. In the fifth chapter “One-Dimensional Well-Posed Nonlocal Elasticity Models for Finite Domains” are introduced. It is shown that some paradoxical results disappear by using the suggested models. In chapter “Iterative Nonlocal Residual Elasticity”, a new approach is presented allowing to avoid some complications of finding solutions of Eringen’s nonlocal model. The seventh chapter deals with “Nonlocal Gradient Mechanics of Elastic Beams Under Torsion”. The theory is applied to nano-electro-mechanical systems. The eighth chapter presents the “Reformulation of the Boundary Value Problems of Nonlocal Type Elasticity: Application to Beams”. In the ninth chapter “Application of Combined Nonlocal and Surface Elasticity Theories to Vibration Response of a Graded Nanobeam” is discussed. It is well known that the correct modeling of small-size structures can be performed if the surface energy is taken into account. Special numerical techniques are necessary for nonlocal theories. In the tenth chapter, “Finite Element Nonlocal Integral Elasticity Approach” is introduced. In the focus of chapter “‘Explicit’ and ‘Implicit’ Non-local Continuum Descriptions: Plate with Circular Hole” is a model for materials with internal material organization when the internal and external length scales are of the same order. The twelfth chapter presents “Micromorphic Continuum Theory: Finite Element Analysis of 3D Elasticity with Applications in Beam- and Plate-Type Structures”. A special 3D micromorphic element with 12 degrees of freedom (3 classical, 9 non-classical) is developed. “Peridynamic modeling of laminated composites” (which is in the focus of chapter “Peridynamic Modeling of Laminated Composites”) presents a new approach to failure analysis. Chapter “Nonlocal Approaches to the Dynamics of Metamaterials” is devoted to a new class of materials with outstanding properties. In the last Chapter “Gradient Extension of Classical Material Models: From Nuclear & Condensed Matter Scales to Earth & Cosmological Scales” the extension of gradient theories to greater sizes is given.

All papers of the present book demonstrate that the modelling of small-size structures cannot be realized using the classical continuum approach. With respect to the scale effects, non-local theories can be applied, but this is not the only one possibility.

Magdeburg, Germany
September 2020

Holm Altenbach
Preface

Classical continuum mechanics has been widely utilized to solve fundamental problems in various fields of engineering and many aspects of physics. Despite its many successes, the classical continuum mechanics cannot always predict well experimentally-observed phenomena in both natural and man-made materials. For example, it fails to describe physical phenomena in which the long-range interactions play a major role. Furthermore, the mathematical modeling of matter via the classical continuum mechanics ignores the fact that the material is made of atoms and so it cannot be directly applied to study the discrete nature of the matter. In addition, the classical continuum mechanics is invariant with respect to length scale and cannot predict the size-dependency of mechanical properties of microstructures and nanostructures.

These limitations have motivated the development of various size-dependent continuum mechanics approaches including micromorphic, micropolar, nonlocal, and high-order strain/rotation gradient mechanics. The conception of these approaches was based on the query, *Is it possible to construct continuum-based approaches that can predict physical phenomena on the micro- or nano-scales?*

The starting point of the development of size-dependent continuum approaches was the monograph of the Cosserat brothers in 1909. Their work was forgotten over half a century since it was ahead of their time. After 1955, the Cosserat theory was extended by several research groups and many advances and criticisms were made since then. In two recent decades, these approaches have been recognized to be practical for mathematical representations of the physical world. In addition, several modifications and improvements of the size-dependent continuum mechanics approaches have been proposed, and their applications to describe the mechanics of various types of advanced materials and structures have been discussed.

The book presents a series of independent chapters written by scientists with worldwide expertise and international reputation in various fields of continuum and computational mechanics, as well as material science. In this book, recent advancements of size-dependent continuum mechanics approaches and their applications to describe the material behavior on different scales have been integrated. One main feature of this book is its in-depth discussions of the vast and
rapidly expanding research works pertinent to the nonlocal continuum mechanics. By compiling different approaches into one book, a unique perspective is provided on the current state of the size-dependent continuum mechanics approaches and what the future holds. It is hoped that the reader will find this book a useful resource as he/she progresses in their study and research.

This book contains fifteen chapters by thirty-five researchers which are from Australia, China, France, Greece, India, Iran, Italy, Qatar, Tunisia, Turkey, UAE, and USA. Chapter “Lattice-Based Nonlocal Elastic Structural Models” describes the lattice-based nonlocal approach and presents some applications of the proposed approach for some lattice structural systems including axial lattices, beam lattices and plate lattices. The contemporary advances in Eringen’s nonlocal elasticity theory with an emphasis on solving structural engineering problems are discussed in chapter “Eringen’s Nonlocal Integral Elasticity and Applications for Structural Models”. In chapter “Nonlocal Mechanics in the Framework of the General Nonlocal Theory”, general nonlocal theory is introduced and it is shown that the theory can be reduced to the strain gradient theory and the couple stress theory. Chapter “Displacement Based Nonlocal Models for Size Effect Simulation in Nanomechanics” considers the displacement based nonlocal models which belong to the mechanically based nonlocality. A well-posed nonlocal differential model for finite domains is developed in chapter “One-Dimensional Well-Posed Nonlocal Elasticity Models for Finite Domains” and its applicability to predict the static behavior of nanorods and nanobeams is investigated. Motivated by the existing complications of finding solutions of Eringen’s nonlocal model, iterative nonlocal residual elasticity is presented in chapter “Iterative Nonlocal Residual Elasticity”.

The nonlocal gradient elasticity theory of inflected nanobeams is extended in chapter “Nonlocal Gradient Mechanics of Elastic Beams Under Torsion” to the mechanics of elastic nanobeams under torsion. Chapter “Reformulation of the Boundary Value Problems of Nonlocal Type Elasticity: Application to Beams” is dedicated to the reformulation of the boundary value problems of nonlocal type elasticity. Application of the combined nonlocal and surface effects on the free and forced vibration response of a graded nanobeam is investigated in Chap. “Application of Combined Nonlocal and Surface Elasticity Theories to Vibration Response of a Graded Nanobeam”.

In chapter “Finite Element Nonlocal Integral Elasticity Approach”, a nonlocal finite element method is developed to study the bending, buckling, and vibration behavior of nanostructures. Chapter “‘Explicit’ and ‘Implicit’ Non-local Continuum Descriptions: Plate with Circular Hole” is focused on the correspondence between “implicit” type Cosserat (micropolar) and “explicit” type Eringen’s two-phase local/nonlocal models, in terms of characteristic quantities. To investigate the mechanical behavior of small-scale structures, a new 12-DOF three-dimensional size-dependent micromorphic element is introduced in chapter “Micromorphic Continuum Theory: Finite Element Analysis of 3D Elasticity with Applications in Beam- and Plate-Type Structures”. Chapter “Peridynamic Modeling of Laminated Composites” presents peridynamic modeling approaches namely bond-based, ordinary state-based, and peridynamic differential operator for predicting
progressive damage in fiber-reinforced composite materials under general loading conditions. Chapter “Nonlocal Approaches to the Dynamics of Metamaterials” provides a concise review of nonlocal theories as applied to metamaterials, with special consideration given to vibrations and dynamics. Finally, in Chapter “Gradient Extension of Classical Material Models: From Nuclear & Condensed Matter Scales to Earth & Cosmological Scales”, a concise review of gradient models (across scales, materials, and processes) is provided based on the internal length gradient approach.

The editors would like to thank all the contributing authors for their participation and cooperation, in spite of their busy work schedules during Covid-19 pandemic, which made this book possible. In addition, we wholeheartedly thank the anonymous reviewers for their carefully performed job and also the team of Springer, especially Dr. Leontina Di Cecco, for their excellent cooperation during the preparation of this edited book. Furthermore, we would like to thank Prof. Holm Altenbach for writing a foreword to the book. Finally, it should be noted that the completion of this book would not have been possible without the help, support and understanding of our families.

Shiraz, Iran
Shiraz, Iran
Naples, Italy
September 2020

Esmaeal Ghavanloo
S. Ahmad Fazelzadeh
Francesco Marotti de Sciarra
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Lattice-Based Nonlocal Elastic Structural Models

Noël Challamel, Chien Ming Wang, Hong Zhang, and Isaac Elishakoff

Abstract This chapter is devoted to lattice-based nonlocal approaches in relation with elastic microstructured elements. Nonlocal continuous approaches are shown to be relevant for capturing length scale effects in discrete structural mechanics models. The chapter contains three complementary parts. In the first part, axial lattices, as already studied by Lagrange during the XVIIIth century are investigated, both for statics and dynamics problems. This discrete model is also called the Born-Kármán lattice model with direct neighbouring interactions. Exact solutions are presented for general boundary conditions. A nonlocal elastic rod model is then constructed from the lattice difference equations. The nonlocal model is similar to the nonlocal model proposed by Eringen in 1983 that is based on a stress gradient approach, although the small length scale of the nonlocal model may differ from statics to dynamics applications. This part is closed with a discussion on generalized lattices with direct and indirect neighbouring interactions and their possible nonlocal modelling. The second part of this study deals with lattice beam elements called Hencky-Bar-Chain models, due to the fact that the discrete beam model was introduced by Hencky in 1920. Exact solutions are presented for general boundary conditions in both statics and dynamics settings. A nonlocal elastic Euler-Bernoulli beam model is then developed from the lattice difference equations. The nonlocal model is similar to a stress
gradient nonlocal Euler-Bernoulli beam, where the nonlocality is of the Eringen type, although the length scale of the nonlocal model may also differ for statics or dynamics applications. The last part is devoted to lattice plates as introduced by Wifi et al. in 1988, and El Naschie in 1990, in connection with the finite difference formulation of Kirchhoff-Love plate models. Exact solutions for lattice plate statics and dynamics problems are presented for the Navier-type boundary conditions. A nonlocal elastic Kirchhoff-Love plate model is then derived based on the difference equations of the lattice plate. The microstructure-based nonlocal model slightly differs from an Eringen stress gradient Kirchhoff-Love plate model. The methodology followed from fundamental lattice microstructures shows that new nonlocal structural elements may be built from physical discrete structural approaches. The nonlocal models derived herein are closely related to the lattice microstructure assumed at the discrete level. It is expected that alternative nonlocal models may be achieved for some other microstructures.

1 Introduction

In this chapter, the behaviour of discrete structural elements (also labelled as microstructured elements or lattice models) is studied both in statics and in dynamics settings. The structural elements considered are typically discrete rods, discrete beams and discrete plates in the elastic range. The discrete models are composed of a finite number of elements connected by some elastic interactions, mainly axial and rotational springs. It is shown that these elastic structural elements possess some scale effects, as compared to their continuous analogues for an asymptotic large number of elements. Exact solutions of lattice structural mechanics problems (including discrete beams and plates) are available in the recent book of Wang et al. [1]. Scale effects in structural mechanics may be also captured within nonlocal mechanics, as extensively developed by Eringen and Kim [2], Eringen [3] (see also the seminal book of Eringen [4]). Owing to some common scale effects, Eringen [3] calibrated a stress gradient model (also called differential nonlocal model) from axial lattice dynamics results. He showed the possibility of approximating some wave dispersive properties of an axial lattice with a nonlocal axial model. This result opens some new directions in the connection of lattice mechanics with nonlocal mechanics. This chapter presents theoretical results in the same direction, for introducing nonlocal rod, beam and plate models from lattice structural mechanics. It is worth mentioning that exact solutions of nonlocal rod, beam and plate models are available in monographs such as the one of Elishakoff et al. [5], Gopalakrishnan and Narendar [6], Karlic et al. [7] or Ghavanloo et al. [8]. However, the connection of these nonlocal models with some discrete lattice formulation has not been studied in details in these books.

Therefore, we will connect the behaviour of lattice structural elements with nonlocal structural mechanics, using phenomenological or lattice-based nonlocal models.
The chapter consists of three complementary parts. In the first part, axial lattices (as already studied by Lagrange) are investigated for some dynamics problems including axial elastic supports. Historically, the theoretical investigation of one-dimensional lattices goes back to the XVIIIth century with the pioneering works of Lagrange. Lagrange [9, 10] calculated the exact eigenfrequencies of finite strings with concentrated masses, which can be viewed as a lattice string. Such a system composed of a finite number of degrees-of-freedom is governed by difference equations, as opposed to continuous systems governed by differential or partial differential equations. The mathematical problem of this difference eigenvalue problem valid for the finite string is in fact equivalent to the vibration problem of a finite microstructured rod in the axial direction or finite shaft in the torsional direction. The discrete axial model composed of masses connected by elastic springs, is also referred to as the Born-Kármán lattice model with direct neighbouring interactions. In this chapter, this Born-Kármán lattice model (or Lagrange model) will be investigated in presence of axial external supports, thus generalizing the results of Lagrange for general boundary conditions and elastic supports. A nonlocal elastic rod model is then developed from the lattice difference equations. The nonlocal model is similar to a stress gradient nonlocal model of Eringen [3], although the length scale of the nonlocal model may differ from statics to dynamics applications. This part ends with a discussion on generalized lattices with direct and indirect neighbouring interactions and their possible nonlocal modelling. The second part deals with lattice beam elements called Hencky-Bar-Chain models [11]. In 1920, Hencky developed a discrete model composed of a finite number of rigid elements connected by rotational springs [11]. Among various discrete structural problems (in-plane buckling of discrete columns, out-of-plane buckling of discrete beams, in-plane buckling of discrete arches and etc.), he solved the buckling problem of this Hencky-Bar-Chain model for some finite number of elements (typically for two, three and four elements). Exact solutions of this buckling problem, whatever the number of elements, have been derived later by Wang [12, 13], from the exact resolution of a linear difference equation. In the chapter, exact solutions of the buckling and vibration of Hencky-Bar-Chain model (labelled as Hencky beam model) are presented for general boundary conditions. A nonlocal elastic Euler-Bernoulli beam model is then built from the lattice difference equations of Hencky-Bar-Chain formulation. The nonlocal model is similar to a stress gradient nonlocal Euler-Bernoulli beam, where the nonlocality is of the Eringen type [3], although the length scale of the nonlocal model may also differ for statics or dynamics applications. The last part is devoted to lattice plates (or microstructured plates) as introduced by Wifi et al. [14] and El Naschie [15], in connection with the finite difference formulation of Kirchhoff-Love plate models. Exact solutions are presented for the Navier-type boundary conditions, any number of elements, both in statics and in dynamics settings. A nonlocal elastic Kirchhoff-Love plate model is then constructed from the lattice difference equations of the plate lattices. The microstructure-based nonlocal model slightly differs from an Eringen stress gradient Kirchhoff-Love plate model. More generally, exact solutions of lattice structural mechanics problems are available in the recent book of Wang et al. [1], including straight, curved Hencky beam models and lattice plate models. Lerbet et
al. [16] also investigated non-conservative discrete structural mechanics problems with circulatory loading. In this chapter, nonlocal rod, beam and plate models are elaborated for each lattice structural model. The methodology followed from fundamental lattice microstructures shows that new nonlocal structural elements may be built from physical discrete structural approaches.

2 Discrete and Nonlocal Rods

2.1 Axial Lattices

This chapter studies the vibration behaviour of axial lattices composed of concentrated masses connected by elastic springs (lattice with direct elastic neighbouring interaction). Exact solutions of this discrete rod problem can be found for general boundary conditions, from the resolution of some linear difference equations. Historically, Lagrange [9, 10] was apparently the first to have derived the exact solutions of fixed-fixed finite lattices. He determined the eigenfrequencies of a lattice string (string with concentrated masses) with fixed-fixed boundary conditions. This problem has been recently revisited by Zhang et al. [17] within the theory of nonlocal string mechanics. From a mathematical point of view, the discrete string problem (or lattice string) is mathematically analogous to the one of a discrete rod (or lattice rod), as considered herein (see also Lagrange [9, 10]). Such a lattice with direct neighbouring interactions may be referred to as a Lagrange lattice or a Born-Kármán lattice [18]. This lattice problem is actually to be solved via a set of linear difference equations with corresponding discrete boundary conditions. The calculation of the eigenfrequencies of such finite lattices is available in many textbooks, for various boundary conditions (see for instance [19–21]). For instance, the vibration frequencies of such a general one-dimensional lattice system were initially calculated by Lagrange [9, 10] for fixed-fixed case (and later by [19], Tong et al. [22], Thomson and Dahleh [20], Blevins [21] or more recently by Challamel et al. [23]). The modelling of the free end boundary condition may be achieved by considering only half of the lumped mass at the lattice border. With such an assumption for the free end condition, the eigenfrequencies of a clamped-free axial lattice has been calculated by Thomson and Dahleh [20] and more recently by Challamel et al. [23]. As detailed in Challamel et al. [24], the lattice string is mathematically equivalent to the axial lattice, but it is also analogous to a torsional lattice, or a shear lattice at some extent. A one-dimensional shear lattice model could capture the shear properties of a multi-storey building (see Thomson and Dahleh [20] or Luongo and Zulli [25] who solved a shear lattice problem with clamped-free ends and assuming a full lumped mass at the border). In this chapter, we will mainly focus on the vibrational behaviour of axial lattices. It is worth mentioning that exact results are also available for the static problem of axial lattices under concentrated load and distributed axial load. For instance, Triantafyllidis and Bardenhagen [26] studied the static behaviours of
a nonlinear axial lattice loaded by a tension at the ends. Hérisson et al. [27] used Hurwitz Zeta functions to derive exact solutions for nonlinear lattices with elastic quadratic interactions and subjected to distributed axial load. Gazis and Wallis [28] derived the exact solutions for an axial lattice of semi-infinite length with direct and indirect neighbouring interactions. The derivation considered linear interaction inside the lattice while except for the lattice boundaries. Charlotte and Truskinovsky [29] studied a four end forces loaded lattice with direct and indirect neighbouring interactions.

Lagrange [9, 10] already in 1759 discussed the link between finite lattice string and the continuous string. The finite string will asymptotically converge to a continuum string when the number of elements increases. The vibration of a continuous string is ruled by a “local” continuous wave equation. The introduction of length scale effects representing the discreteness of the lattice structure in a corrected (or enriched) continuous model is more recent. The development of higher-order continuous models from the lattice formulation may be achieved by expanding the pseudo-differential operators with high-order continuous differential operators. This methodology, also called continualization technique, has been initiated in the 1960s especially for applications in the field of discrete wave equations, or wave in nonlinear axial lattices (see for instance Kruskal and Zabusky [30]). Various approximations of the pseudo-differential operators may be used, using power series (Taylor expansion) or rational series (Padé approximants). Enriched continua, also called quasicontinua by Collins [31] can be viewed as a continuous approximation of the exact (or reference) lattice problem. They mathematically differ from the lattice problem, due to the asymptotic expansion of the pseudo-differential operators at a given order. For instance, Kruskal and Zabusky [30] expanded the pseudo-differential operator involved in a nonlinear axial lattice in power series, by considering a fourth-order Taylor-based asymptotic expansion. It can be shown that the additional term responsible of small length scale effects due to this expansion modifies the potential energy functional, which is then no more positive definite. Some alternative expansions of the pseudodifferential operators have been used, for deriving alternative quasicontinua. For instance, Benjamin et al. [32] transformed Korteweg-de Vries wave equation by replacing higher-order spatial derivatives with coupled spatio-temporal derivatives. Collins [31] and Rosenau [33] used the same methodology to avoid higher-order uncoupled spatial derivatives, by inverting the spatial pseudo-differential operator. For linear elastic interactions, Jaberolanssar and Peddieson [34] obtained a nonlocal wave equation with coupled spatio-temporal derivatives (without higher-order spatial derivatives). The result has been generalized by Rosenau [33] who expanded the pseudo-differential operator of the nonlinear lattice, with a Padé approximant, for deriving a consistent nonlocal (and nonlinear) wave equation without higher-order spatial derivatives. Jaberolanssar and Peddieson [34] and Rosenau [33] obtained a nonlocal wave equation built from the difference equations of the axial elastic lattice. Eringen [3] postulated a nonlocal elastic model from a phenomenological point of view, in a differential format [3, 4]. The stress gradient of Eringen [3] relates the stress and the strain in an implicit differential form. The obtained nonlocal wave equation is mathematically similar to the one issued of a continualization process applied to the difference equations.
of the axial lattice, as followed by Jaberolanssar and Peddieson [34] and Rosenau [33]. Therefore, there is a strong relation between the discrete lattice mechanics theory and the continuous nonlocal mechanics theory, where the nonlocal terms of the approximated continuous model can be calibrated with respect to the lattice spacing. Enriched continuous formulations of axial lattices have been already derived in dynamics [3, 33] as well as in statics applications. For instance, Triantafyllidis and Bardenhagen [26] used a Taylor expansion of the pseudo-differential operators (continualization technique) to derive a gradient elasticity theory applied to the static response of an axial lattice. Hérisson et al. [27] used Padé approximant for characterizing nonlinear lattices with a nonlocal nonlinear continuous bar model. Gul et al. [35] also used a Taylor expansion of the pseudo-differential operator to approximate the vibration behavior of finite axial lattices with a gradient elasticity model (with no definite positive energy functional).

It has been already commented that the continualization technique applied to the vibration of the lattice problem could be reformulated in terms of nonlocal wave equation. The continuous nonlocal phenomenological model of Eringen [3] (stress gradient model) has been identified based on the wave dispersive properties of the axial lattice. Aydogdu [36] calculated the analytical vibration frequencies of finite rods with clamped-clamped and clamped-free ends based on Eringen’s nonlocal theory. A hybrid gradient/nonlocal model (which includes the strain gradient elasticity model and the stress gradient model of Eringen) has been developed by Challamel et al. [37], in order to better calibrate the dispersive parameters of the linear axial lattice. Aydogdu [38] generalized his earlier results [36] by considering a nonlocal elastic rod embedded in elastic medium (with linear elastic interaction with the substrate). The nonlocal longitudinal wave propagation problem for multi-walled carbon nanotubes with van der Waals interactions between each nanotube walls has been studied by Aydogdu [39].

This chapter is mainly focused on the axial lattice problem. Axial lattice with fixed-fixed and fixed-free boundary conditions in the presence of elastic neighbouring interactions are analyzed in statics and dynamics. It should be noted that elastic interactions on an elastic substrate are also included in Rosenau [33] as we shall study in this chapter. Also, a nonlocal model will be continualized from the discrete equations to fit the behavior of an axial lattice on elastic support (see also Challamel et al. [40]).

### 2.2 Lattice Formulation: Governing Equations

In the first part of this chapter, we study a one-dimensional uniform lattice composed of \( n + 1 \) masses connected by \( n \) springs with identical stiffness as shown in Fig. 1.

The mass \( m_i \) of each particle are identical, except for the border mass which is \( m_i/2 \). \( L \) is the total length of the lattice system and \( L = n \times a \), where \( a \) is the nodal spacing in the lattice. Fixed-fixed and fixed-free boundary conditions are specifically studied. For fixed-fixed boundary conditions, there are \( n + 1 \) particles with two of
them attached at the border, i.e. \( n - 1 \) free particles, whereas for the fixed-free chain, there are \( n + 1 \) particles, one of them attached, and the other free i.e. \( n \) free particles.

The balance equations of the axial lattice with only direct neighbouring interactions including the presence of distributed load can be obtained from:

\[
\frac{N_{i+1/2} - N_{i-1/2}}{a} = \rho A \frac{d^2 u_i}{dt^2} + ku_i - q_i
\]  

(1)

where \( k \) is the equivalent stiffness of the elastic substrate, \( \rho A \) denotes the mass density per unit length, \( q_i \) is the distributed axial load applied at node \( i \), \( u_i \) is the displacement of node \( i \) and \( N_i \) denotes the normal force in the \( i \)th spring (with half a
shift for the considered element). The mass distribution is uniform \( m_i = \rho A a \) except at the border where \( m_0 = m_n = \rho A a / 2 \). Equations are derived for a one-dimensional axial lattice, but the developments could also be used for string lattices, shear lattices and torsional lattices (see Ref. [23] or Ref. [24] for a discussion on the analogies between these different kinds of lattices). We shall mainly focus on the calculation of eigenfrequencies of this lattice. The distributed axial force \( q_i \) is assumed to vanish, i.e. \( q_i = 0 \) (see Challamel et al. [40] for the general calculations valid in case of parabolic axial forces).

The normal force in the spring \( i + 1/2 \) (defined in the element) is relative to the axial displacement between two adjacent nodes:

\[
N_{i+1/2} = EA \frac{u_{i+1} - u_i}{a}
\]

where \( EA/a \) is the stiffness of the axial spring. The governing mixed functional differential equation (or mixed differential-difference equation – see Myshkis [41]) is obtained by substituting Eq. (2) into Eq. (1):

\[
EA \frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} - \rho A \frac{d^2 u_i}{dt^2} - ku_i = -q_i
\]

In this chapter, we will focus on the dynamic analysis and will not take into account the distributed axial forces so that \( q_i = 0 \). Distributed forces and elastic medium are absent in the usual Born-von Kármán lattice equations, i.e. for \( q_i = 0 \) and \( k = 0 \) so that the governing Eq. (3) becomes

\[
q_i = k = 0 \Rightarrow EA \frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} - \rho A \frac{d^2 u_i}{dt^2} = 0
\]

The dynamic behaviours of this lattice will be studied for two archetypal boundary conditions, namely for fixed-fixed and fixed-free boundary restraints. The boundary conditions for fixed-fixed ends (as shown in Fig. 1) are given by:

\[
u_0 = 0 \quad \text{and} \quad u_n = 0
\]

whereas the boundary conditions for fixed-free ends should be written as:

\[
u_0 = 0 \quad \text{and} \quad \frac{N_{n+1/2} - N_{n-1/2}}{a} = \frac{\rho A}{2} \frac{d^2 u_n}{dt^2} + \frac{k}{2} u_n \quad \text{with} \quad N_{n+1/2} = 0
\]

which may be equivalently formulated as

\[
u_0 = 0 \quad \text{and} \quad EA \frac{u_n - u_{n-1}}{a} = -a \frac{\rho A}{2} \frac{d^2 u_n}{dt^2} - a \frac{k}{2} u_n
\]
Note that half mass is attached at the last spring of half elastic interaction. As notably mentioned by Maugin [42] (see more recently Challamel et al. [24] for structural mechanics applications), the difference equations valid for lattice elasticity exactly corresponds to the finite difference approximation for the continuous medium (elastic medium). Hence, the theoretical foundation of lattice mechanics is firmly related to the numerical analysis of continuum elasticity as solved by the finite difference method. The free boundary condition considered is related to the central finite difference discretization of the continuous equations. The last mass \( m_n = m_i/2 \) naturally appears in this scheme, as shown by LeVeque [43] who treated Neumann type continuous boundary conditions by using a central finite difference approximation. Challamel et al. [23] also considered this kind of discrete boundary condition for formulating the Neumann boundary condition. Of course, it would have been possible to get an exact solution of the lattice problem with a mass \( m_n \) equal to the other values \( m_i \) for instance (or a border mass not equal to \( m_i/2 \)), but the resulting discrete solution would be slightly different from the solution presented herein. In addition, Kivshar et al. [44] set last particle mass \( m_n \) different from half of the internal particle mass and then the associated continuous boundary conditions are in a mixed-type in presence of both displacement and its derivative.

It is possible, equivalently, to derive the governing difference Eq. (3) from energy arguments, based on the potential energy \( W \) given by

\[
W = \sum_{i=0}^{n-1} \frac{EA}{2} a \left[ \left( \frac{u_{i+1} - u_i}{a} \right)^2 \right] + \sum_{i=1}^{n-1} \frac{k}{2} au_i^2 + \frac{k}{4} au_0^2 + \frac{k}{4} au_n^2 \tag{8}
\]

With half of the mass at the ends, the kinetic energy \( T \) could be written as

\[
T = \sum_{i=1}^{n-1} \frac{\rho A}{2} a \left( \frac{du_i}{dt} \right)^2 + \frac{\rho A}{4} a \left( \frac{du_0}{dt} \right)^2 + \frac{\rho A}{4} a \left( \frac{du_n}{dt} \right)^2 \tag{9}
\]

By using Hamilton’s principle, the mixed differential-difference governing equation is found again. Considering a harmonic motion \( u_i(t) = u_i e^{j\omega t} \) where \( j = \sqrt{-1} \) and \( \omega \) is the angular frequency of vibration, the linear mixed differential-difference Eq. (3) may reduce in a linear second-order difference equation in space:

\[
EA \frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} + \left( \rho A \omega^2 - k \right) u_i = 0 \tag{10}
\]

### 2.3 Lattice Formulation: Resolution

By introducing the following dimensionless parameters

\[
q^* = \frac{qL^2}{EA}, \quad k^* = \frac{kL^2}{EA}, \quad \beta = \rho \frac{\omega^2 L^2}{E} \tag{11}
\]
the linear second-order difference equation can be expressed as:

\[ u_{i+1} - \left(2 + \frac{k^n - \beta}{n^2}\right) u_i + u_{i-1} = 0 \]  

(12)

This linear difference equation can be solved by assuming a displacement solution in a form of power function (Goldberg [45]):

\[ u_i = u_0 \lambda^i \]  

(13)

The substitution of Eq. (13) into the difference Eq. (12) furnishes the following auxiliary equation:

\[ \lambda^2 - \left(2 + \frac{k^n - \beta}{n^2}\right) \lambda + 1 = 0 \]  

(14)

Note that this equation is palindromic, implying that if \( \lambda \) is its solution, so is its reciprocal \( 1/\lambda \).

The auxiliary equation admits two complex conjugate solutions

\[ \lambda = 1 - \frac{\beta - k^n}{2n^2} \pm \sqrt{\left(1 - \frac{\beta - k^n}{2n^2}\right)^2 - 1} = \cos \phi \pm j \sin \phi \]

with \( \cos \phi = 1 - \frac{\beta - k^n}{2n^2} \)  

(15)

The general solution for the linear difference Eq. (12) can finally be expressed in trigonometric functions:

\[ u_i = A \cos (\phi i) + B \sin (\phi i) \]  

(16)

For fixed-fixed boundary conditions, using one boundary condition \( u_0 = 0 \) necessarily shows that the eigenmode is reduced to a simple sinusoidal function:

\[ u_i = B \sin (\phi i) \]  

(17)

The eigenfrequencies are obtained by using the second boundary condition \( u_n = 0 \) and consequently the frequency equation is given by

\[ \sin (n\phi) = 0 \text{ with } \phi = \arccos \left[1 - \frac{\beta - k^n}{2n^2}\right] \]  

(18)

The exact eigenfrequencies of the lattice system with fixed-fixed boundary conditions are obtained for the \( m \)th mode from Eq. (18):

\[ \beta_{m,n} = 4n^2 \sin^2 \left(\frac{m \pi}{2n}\right) + k^n \text{ for } m \in \{1, 2, ..., n - 1, n\} \]  

(19)
For fixed-free ends, the two boundary conditions are given by the kinematic constraint and the free end boundary condition \( u_n - u_{n-1} = (\beta - k^*) u_n / 2n^2 \). The consideration of the essential boundary condition \( u_0 = 0 \) leads again to the trigonometric solution (17). One obtains from the second boundary condition:

\[
\cos(n\phi) = 0 \quad \text{with} \quad \phi = \arccos \left[ 1 - \frac{\beta - k^*}{2n^2} \right] \tag{20}
\]

The natural frequencies of the \( m \)th mode, valid for fixed-free boundary conditions are finally obtained from Eq. (20):

\[
\beta_{m,n} = 4n^2 \sin^2 \left[ \frac{(2m - 1) \pi}{4n} \right] + k^* \quad \text{for} \quad m \in \{1, 2, \ldots, n - 1, n\} \tag{21}
\]

Both formula valid for fixed-fixed and fixed-free boundary conditions coincide with the ones presented in Challamel et al. [23] without considering the external medium, i.e. for \( k^* = 0 \).

2.4 Nonlocal Continualized Model and Eringen’s Model

Following the methodology introduced by Kruskal and Zabusky [30] for nonlinear lattices, the mixed differential-difference Eq. (3) is continualized to:

\[
EA \left[ \frac{u(x + a, t) - 2u(x, t) + u(x - a, t)}{a^2} \right] - \rho A \frac{\partial^2 u(x, t)}{\partial t^2} - ku(x, t) = -q(x) \tag{22}
\]

It is possible to expand the spatial difference operator in Taylor series for the spatial difference operator, for sufficiently smooth displacement fields (see Salvadori [46], Kruskal and Zabusky [30], Rosenau [33], Andrianov and Awrejcewicz [47, 48] and Andrianov et al. [49]):

\[
u(x \pm a, t) = \sum_{k=0}^{\infty} \frac{(\pm a)^k}{k!} \frac{\partial^k}{\partial x^k} u(x, t) = \left[ e^{a\partial/\partial x} \right] u(x, t) \tag{23}\]

so that Eq. (22) could also be rewritten by using a pseudo-differential operator:

\[
\frac{4EA}{a^2} \left[ \sinh^2 \left( \frac{a \partial}{2 \partial x} \right) \right] u(x, t) - \rho A \frac{\partial^2 u(x, t)}{\partial t^2} - ku(x, t) = -q(x) \tag{24}
\]

The pseudo-differential operator can be truncated at order 4, by using a Taylor-based expansion [30]:
\[
\frac{4EA}{a^2} \left[ \sinh^2 \left( \frac{a}{2} \frac{\partial}{\partial x} \right) \right] u(x, t) = EA \frac{\partial^2}{\partial x^2} \left[ 1 + \frac{a^2}{12} \frac{\partial^2}{\partial x^2} \right] u(x, t) + O(a^4) \quad (25)
\]

Rosenau [33], following the idea of Benjamin et al. [32] or Collins [31], replace the higher-order spatial derivatives using Padé approximant of order [2, 2] to expand the pseudo-differential operator (see Baker and Graves-Morris [50] for an extensive analysis of Padé approximants):

\[
\frac{4EA}{a^2} \left[ \sinh^2 \left( \frac{a}{2} \frac{\partial}{\partial x} \right) \right] u(x, t) = EA \frac{\partial^2}{\partial x^2} u(x, t) + \ldots \quad (26)
\]

The pseudo-differential equation would then be written with coupled spatio-temporal derivatives based on Padé approximant:

\[
EA \frac{\partial^2 u(x, t)}{\partial x^2} - \rho A \left[ 1 - \frac{a^2}{12} \frac{\partial^2}{\partial x^2} \right] \frac{\partial^2 u(x, t)}{\partial t^2} - k \left[ 1 - \frac{a^2}{12} \frac{\partial^2}{\partial x^2} \right] u(x, t) = - \left[ 1 - \frac{a^2}{12} \frac{\partial^2}{\partial x^2} \right] q(x) \quad (27)
\]

Jaberolanssar and Peddieson [34] obtained Eq. (27) in the context of dynamics and without considering elastic medium interaction, namely for \( q(x) = 0 \) and \( k = 0 \). Rosenau [33] derived a similar equation in a dynamic context, i.e. for \( q(x) = 0 \), in the presence of elastic external interaction.

It is possible to derive equivalently the governing partial differential Eq. (27) from the following total potential energy

\[
W = \int_0^L \left( \frac{1}{2} EA \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} ku^2 + \frac{1}{2} l_c^2 k \left( \frac{\partial u}{\partial x} \right)^2 - qu - l_c^2 \left( \frac{dq}{dx} \right) \left( \frac{\partial u}{\partial x} \right) \right) dx
\]

with \( l_c^2 = \frac{a^2}{12} \) \quad (28)

coupled with the modified kinetic energy:

\[
T = \int_0^L \left( \frac{1}{2} \rho A \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} l_c^2 \rho A \left( \frac{\partial^2 u}{\partial x \partial t} \right)^2 \right) dx \quad (29)
\]

This modified kinetic energy which includes additional small length scale terms, has been obtained equivalently by Rosenau [51] from continualizing the lattice kinetic energy. Rosenau [51] also derived the modified energy potential from the continualization of the lattice potential energy including the elastic medium interaction while without distributed forces.
Furthermore, it is possible to re-express Eq. (27) from the following nonlocal (differential) constitutive law, coupled to local balance equations:

\[
\left[ 1 - \frac{a^2}{12} \frac{\partial^2}{\partial x^2} \right] N(x,t) = EA \frac{\partial u}{\partial x} (x,t)
\]

and

\[
\frac{\partial N(x,t)}{\partial x} = \rho A \frac{\partial^2 u(x,t)}{\partial t^2} + ku(x,t) - q(x) \quad (30)
\]

One easily recognizes the stress gradient model of Eringen [3] in Eq. (30) where the length scale \(l_c\) of the nonlocal Eringen’s model, is calibrated with respect to the lattice spacing \(a\):

\[
N - l_c^2 \frac{\partial^2 N}{\partial x^2} = EA \frac{\partial u}{\partial x} \quad \text{with} \quad l_c^2 = (e_0 a)^2 = \frac{a^2}{12} \quad \text{and} \quad e_0 = \frac{1}{2\sqrt{3}} \quad (31)
\]

It is shown from Eq. (31) that the small length scale coefficient of the nonlocal model can be calibrated from the lattice problem and is found to be \(e_0 = 1/2\sqrt{3}\). The nonlocal model derived from the lattice equations exactly coincides with the nonlocal Eringen’s stress gradient model, where the small length scale parameter of the nonlocal model is calculated regarding the nodal spacing in lattice. Eringen’s nonlocal model [3] and the lattice-based nonlocal model (as obtained by Jaberolanssar and Peddieson [34] or Rosenau [33]) are both formulated in a modified wave equation where the additional small length scale term uses coupled second-order spatial and time derivatives. Similar models with additional spatio-temporal coupling terms have been proposed for other applications of one-dimensional continuum system by Love [52] for axial vibration problems in the presence of lateral inertia effects, Mindlin [53] and Polyzos and Fotiadis [54] for axial vibration problems, and by Rayleigh [55] for the bending problem of beam including the effect of rotary inertia.

It is possible to extract the normal force (of Eringen’s stress gradient model) from Eqs. (30) and (31), thus leading to:

\[
N = \left( EA + kl_c^2 \right) \frac{\partial u}{\partial x} + l_c^2 \rho A \frac{\partial^3 u}{\partial x \partial t^2} - l_c^2 \frac{dq}{dx} \quad (32)
\]

This expression of normal force corresponds to the natural boundary condition derived from application of Hamilton’s principle. The nonlocal boundary condition raised from potential energy Eq. (28) and kinetic energy formula Eq. (29) is given by

\[
\left[ (EA + kl_c^2) \frac{\partial u}{\partial x} + l_c^2 \rho A \frac{\partial^3 u}{\partial x \partial t^2} - l_c^2 \frac{dq}{dx} \right] \delta u \bigg|_0^L = 0 \quad (33)
\]

It is shown from Eq. (33), that no additional boundary condition is required for such a nonlocal model, as opposed to gradient elasticity models based on higher-order differential equations with additional extra boundary conditions.