Outstanding Contributions to Logic 19

Judit Madarász
Gergely Székely *Editors*

Hajnal Andréka and István Németi on Unity of Science

From Computing to Relativity Theory Through Algebraic Logic

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Outstanding Contributions to Logic

Volume 19

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Preface

This volume has been prepared in appreciation of the truly outstanding contributions of Hajnal Andréka and István Németi to logic. One who does not know them or their joint careers and working style may wonder why this is a joint volume, but for anyone who knows them well or who reads in their joint autobiography how their careers have been closely intertwined since at least as far back as 1971, it should be clear that having one volume is a natural choice.

Not only do they write most of their publications jointly, but they also give conference talks and university lectures together. Their talks are like a kind of dance. They frequently switch roles spontaneously while speaking, even during the course of a single talk, flawlessly and smoothly complementing each other's trains of thought. They are almost always together. One rarely sees one of them without the other. Somebody once said about them that "they are like a single soul in two bodies" and we have to admit it is quite an accurate description!

We are both collaborators and former students of Hajnal and István. However, they are more than mere teachers and collaborators for us. They took us under their wings as undergraduate students and continued advising and supervising us during graduate school and even as postgraduates. They treated us, just like all of their students, like family members. They often invited us to their home to work and discuss ideas sitting at the round table in their living room. These were round-table discussions in every sense, in which everyone was treated equally and ideas were welcomed from every participant. For us, they are also mentors and good friends: preparing this volume has been a great honor for us. We are truly grateful to them for their continuous support and the so many seeds of insight they have planted in our minds.

For Hajnal and István, looking for deeper understandings and connecting different research areas has always played a central role. Therefore, we have invited researchers working in various areas related to logic to write chapters of this book representing the research-area-integrating and insight-seeking spirit of Hajnal and István's work.

This volume contains 19 invited chapters organized into 3 parts: Computing, Algebraic Logic, and Relativity Theory, corresponding, in chronological order, to the 3 main areas of research where Hajnal and István have contributed the most. In the autobiography of Hajnal and István at the end of this volume, one can find how these areas are interconnected in their research.

We are grateful to the series editor Sven Ove Hansson and the whole Editorial Board of this series for the opportunity to publish this volume in the series Outstanding Contributions to Logic. We are also grateful to Ram Prasad Chandrasekar and Christi Lue for their help and patience.

Budapest, Hungary
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Judit Madarász
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Part I
Computing

Chapter 1
Algebraic Logic and Knowledge Bases

Elena Aladova, Boris Plotkin, and Tatjana Plotkin

Abstract Knowledge bases theory provides an important example of the field where applications of universal algebra and algebraic logic look very natural, and their interaction with practical problems arising in computer science might be very productive. In this paper we study the equivalence problem for knowledge bases. Our interest is to find out how the informational equivalence is related to the logical description of knowledge. The main objectives of this paper are logically-geometrically equivalent and LG-isotypic knowledge bases. We will see that these notions give us a good characterization of knowledge bases.

Keywords Algebraic logic · Knowledge base · Halmos algebras · Galois correspondence · Syntax · Semantics · Equivalence of models · Equivalence of knowledge bases · Category · Ultrafilter

1.1 Introduction

Knowledge bases and databases theories provide an important example of the field where applications of universal algebra and algebraic logic look very natural. Historically, theoretical research in databases started at the beginning of sixties. The classical work of Codd [7] laid foundations of relational databases. Since then,advances in

To Hajnal Andréka and István Németi with admiration.

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database theory are tightly related to mathematical logic, theory of algorithms and general algebra.

Plotkin in [16] proposed a mathematical model of a database and gave a formal definition of the databases equivalence concept. As a result, the algebraic model of a knowledge base has been introduced and developed, see [16, 17, 19]. We should emphasize that the model of a knowledge base under consideration involves various ideas of universal algebra, algebraic logic and algebraic geometry. The main peculiarity of this approach is that a database and a knowledge base are treated as certain algebraic and model-theoretical structures.

There are a lot of specialized knowledge bases and it is desirable to determine their characteristic properties without referring to their complicated structure and without studying their detailed architecture. In this concern, *our aim is to distinguish logical invariants of knowledge bases which rigidly determine them*. The selection of logical characterizations of a knowledge base is motivated by the knowledge base model in question. We consider queries to a knowledge base as sentences in some logical language, and in order to operate with them properly one should attract logical tools. Moreover, we convert logic into algebra. This converting allows us to study syntactical logical notions and questions by using more transparent algebraic and geometric structures, i.e., via semantics.

Our main tool is Halmos algebras which were presented under the name of polyadic algebras in [9], and introduced in [16] for the multi-sorted case and an arbitrary variety of algebras. These generalizations are important for the logical geometry and databases applications.

In plain words, Halmos algebras are intended to replace the logical system by an algebraic system, equivalent to the original logic. This way of thinking is well-known for Boolean algebras: to think about propositional sentences as elements in a Boolean algebra is a folklore. The principal goal of algebraization of a first order logic is to do essentially the same for the logic enriched by quantifiers and predicates. Thus, the appropriate (in the sense of algebraization of a first order logic) algebra should be a Boolean algebra, equipped with quantifiers (defined in an abstract way as operations) and operations of replacement of variables. This requires additional axioms and the resulting algebraic object is called a Halmos algebra.

Note here, that there are other algebraizations of the first order logic, see, for example, [5, 6, 8, 10]. A survey of algebraizations of quantifier logics is [15]. For some reason we have chosen Halmos algebras as a tool, appropriate for our goals.

In this paper we discuss equivalence problem for knowledge bases. We focus our attention on a special kind of equivalence, namely, *informational equivalence*. Informally, one can say that two knowledge bases are informationally equivalent if and only if all information that can be retrieved from one knowledge base can be also obtained from the other one and vice versa. *Our main goal is to find out how the informational equivalence is related to the logical description of a knowledge base*.

1.2 Preliminaries

In this Section we present the main apparatus, which is needed further. We give a brief review of basic notions and notations from universal algebraic geometry. In particular, we define Halmos algebras and Halmos categories, and construct the Galois correspondence which plays an important role in all considerations. The material of this section can be found in [17, 19], see also [2, 20, 21].

1.2.1 Basic Notions and Notations

Let $X^0 = \{x_1, \ldots, x_n, \ldots\}$ be an infinite set of variables. Denote by Γ the collection of all finite subsets $X = \{x_{i_1}, \ldots, x_{i_k}\}$ of X^0.

Let Θ be a *variety of algebras*, that is, a class of algebras satisfying a set of identities. We denote by $Var(H)$ the variety generated by the algebra H.

Denote by $W(X)$ the *free algebra* in the variety Θ with free generating set X, $X \in \Gamma$. All free algebras $W(X) \in \Theta$ form a *category of free algebras* Θ^0 with homomorphisms $s : W(X) \to W(Y)$ as morphisms, $X, Y \in \Gamma$.

By a *model* $\mathscr{H}$ we mean a triple (H, Ψ, f), where H is an algebra from Θ, Ψ is a set of relation symbols φ, f is an interpretation of all φ in H (see, for instance, [13, 16]).

Let H be an algebra in Θ and $X = \{x_1, \ldots, x_n\}$. A *point* $(a_1, \ldots, a_n)$ from n-th Cartesian power of H can be represented as a map $\mu : X \to H$ such that $a_i = \mu(x_i)$. This map can be extended up to homomorphism of algebras $\mu : W(X) \to H$. Thus, the point $(a_1, \ldots, a_n)$ can be also viewed as a homomorphism $\mu : W(X) \to H$.

Denote by $Hom(W(X), H)$ the set of all homomorphisms from $W(X)$ to H. We will regard $Hom(W(X), H)$ as an *affine space*. All affine spaces $Hom(W(X), H)$ with various $X \in \Gamma$ constitute the *category* $\Theta^0(H)$ *of affine spaces* with morphisms $\widetilde{s} : Hom(W(X), H) \to Hom(W(Y), H)$, for each homomorphism of free algebras $s : W(Y) \to W(X)$. The map $\widetilde{s}$ is defined as $\widetilde{s}(\mu) = \mu s$, where $\mu : W(X) \to H$, $\widetilde{s}(\mu) : W(Y) \to H$.

Categories Θ^0 and $\Theta^0(H)$ are very important in what follows. Moreover, it is known [14] that the categories Θ^0 and $\Theta^0(H)$ are contravariant isomorphic if and only if $Var(H) = \Theta$.

1.2.2 Halmos Categories and Halmos Algebras

Halmos categories were introduced in [17, 19]. They are related to the first order logic in a way analogous to the relationship between Boolean algebras and propositional logic. Such an approach allows us to use techniques and structures of algebraic logic (see [9, 16]). The immediate advantage of this phenomenon is that we can view queries to a knowledge base and replies to these queries as objects of the same nature, i.e., objects of Halmos categories. Then the transition query-reply can be treated as a functor (for details see Sect. 1.3).

1.2.2.1 Extended Boolean Algebras

We start from the notion of an existential quantifier on a Boolean algebra. Let B be a Boolean algebra. Existential quantifier on B is an unary operation $\exists : B \to B$ such that the following conditions hold:

1. $\exists\, 0 = 0$,
2. $a \leq \exists a$,
3. $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

Universal quantifier $\forall : B \to B$ is dual to $\exists : B \to B$, they are related by $\forall a = \neg(\exists(\neg a))$.

Let Θ be an arbitrary variety of algebras and $W(X)$ the free algebra in Θ with the free generating set X (i.e., Θ can be variety of groups, of associative algebras, etc.).

Definition 1 Let a set of variables $X = \{x_1, \ldots, x_n\}$ and a set of relations Ψ be given. A Boolean algebra B is called an extended Boolean algebra over $W(X)$ relative to Ψ, if

1. the existential quantifier $\exists x$ is defined on B for all $x \in X$, and $\exists x \exists y = \exists y \exists x$ for all $x, y \in X$;

2. to every relation symbol $\varphi \in \Psi$ of arity n_φ and a collection of elements $w_1, \ldots, w_{n_\varphi}$ from $W(X)$ there corresponds a nullary operation (a constant) of the form $\varphi(w_1, \ldots, w_{n_\varphi})$ in B.

Thus, the signature L_X of an extended Boolean algebra consists of the Boolean connectives, existential quantifiers $\exists x$ and constants $\varphi(w_1, \ldots, w_{n_\varphi})$:

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\},$$

where M_X is a set of all $\varphi(w_1, \ldots, w_{n_\varphi})$.

We provide two important examples of extended Boolean algebras.

Example 1 Let a model $\mathscr{H} = (H, \Psi, f)$ be given. Let $Bool(W(X), H)$ be the Boolean algebra of all subsets in $Hom(W(X), H)$. One can equip the Boolean algebra $Bool(W(X), H)$ with the structure of an extended Boolean algebra (cf. [16]). First, we define quantifiers $\exists x$, $x \in X$, on $Bool(W(X), H)$. We set $\mu \in \exists x A$ if and only if there exists $\nu \in A$ such that $\mu(y) = \nu(y)$ for every $y \in X$, $y \neq x$. It can be checked that $\exists x$ defined in such a way is, indeed, an existential quantifier.

According to Definition 1, given a relation symbol $\varphi \in \Psi$ of arity m and a collection of elements $w_1, \ldots, w_m$ from $W(X)$ we shall define a constant in

$Bool(W(X), H)$. Denote this constant by $[\varphi(w_1, \dots, w_m)]_{\mathscr{H}}$ and define it as a subset in $Bool(W(X), H)$ consisting of all points $\mu : W(X) \to H$ satisfying relation $\varphi(w_1, \dots, w_m)$.

Thus, the Boolean algebra $Bool(W(X), H)$ is equipped with the structure of an extended Boolean algebra. We denote this extended Boolean algebra by $Hal_{\Theta}^{X}(\mathscr{H})$.

Example 2 Another important example of an extended Boolean algebra is presented by the algebra of formulas $\Phi_0(X) = \mathfrak{L}_X/\tau_X$, where $\mathfrak{L}_X$ is the absolutely free algebra in the signature L_X over the set M_X, τ_X is a congruence relation (Lindenbaum–Tarski congruence) on $\mathfrak{L}_X$ defined by the rule: $u\tau_X v$ if and only if $\vdash (u \to v) \wedge (v \to u)$, $u, v \in \mathfrak{L}_X$. Boolean operations and quantifiers on $\Phi_0(X)$ are naturally inherited from $\mathfrak{L}_X$. For more details see [1, 2, 17].

1.2.2.2 Halmos Categories

Now we define *a Halmos category* which plays a crucial role in our approach (cf. [16, 17, 20]).

Definition 2 A category $\mathbb{H}$ is a Halmos category if:

1. Its every object has the form $\mathbb{H}(X)$, where $\mathbb{H}(X)$ is an extended Boolean algebra over $W(X)$.
2. The morphisms in $\mathbb{H}$ correspond to morphisms in the category Θ^0. To every morphism $s : W(X) \to W(Y)$ in Θ^0 it corresponds a morphism $s_* : \mathbb{H}(X) \to \mathbb{H}(Y)$ in $\mathbb{H}$ such that

   a. the transitions $W(X) \to \mathbb{H}(X)$ and $s \to s_*$ determine a covariant functor from Θ^0 to $\mathbb{H}$.
   b. $s_* : \mathbb{H}(X) \to \mathbb{H}(Y)$ is a homomorphism of corresponding Boolean algebras.

3. Let homomorphisms s, s^1, s^2: $W(X) \to W(Y)$ and corresponding morphisms s_*, s_*^1, s_*^2: $\mathbb{H}(X) \to \mathbb{H}(Y)$ be given. Let $a \in \mathbb{H}(X)$. The identities controlling the interaction of morphisms with quantifiers are as follows:

   3.1. $s_*^1 \exists x a = s_*^2 \exists x a$, if $s_1(x') = s_2(x')$ for every $x' \in X$ such that $x' \neq x$;
   3.2. $s_* \exists x a = \exists (s(x))(s_* a)$, if $s(x) = y$ and y is a variable which does not belong to the support of $s(x')$, for every $x' \in X$ such that $x' \neq x$. For the definition of support see [16]. In plain words, this condition means that y does not participate in the shortest expression of the element $s(x') \in W(Y)$ through the elements of Y.

4. Let φ be a relation symbol from Ψ of arity n_φ, $w_1, \dots, w_{n_\varphi}$ be a collection of elements from $W(X)$ and $\varphi(w_1, \dots, w_{n_\varphi})$ be the corresponding constant in $\mathbb{H}(X)$. The identities controlling the interaction of morphisms with constants are as follows:

$$s_*(\varphi(w_1, \dots, w_{n_\varphi})) = \varphi(s(w_1), \dots, s(w_{n_\varphi})),$$

where $s : W(X) \to W(Y)$ is a homomorphism of free algebras and $s_*\colon \mathbb{H}(X) \to \mathbb{H}(Y)$ is the corresponding morphisms.

Let us give some explanations concerning Definition 2. Axioms (*3.1*) and (*3.2*) which look strange are not so complicated as they seem at first glance. They accumulate usual and intuitively clear relations between quantifications and transformations of variables in first order logic calculus. For example, axiom (*3.1*) says that homomorphisms s^1 and s^2 are identity maps on $X \setminus \{x\}$. This corresponds to the familiar fact that it is possible to replace a quantified variable in a formula by another one. Axiom (*3.2*) says that if the replaced variable $y = s(x)$ does not participate in the support of $s(x') \in W(Y)$ for all $x' \in X$ different from x, then it is possible to quantify y.

Suppose that the set of relation symbols Ψ contains the symbol of equality predicate "$\equiv$". Let B be an extended Boolean algebra over $W(X)$ with respect to Ψ. One can define an *equality* on B as a binary predicate

$$\equiv\colon W(X) \times W(X) \to B,$$

which takes a pair $w, w' \in W(X)$ to the constant $w \equiv w'$ in B subject to conditions:

1. $\big(w_1 \equiv w_1'\big) \le \big(w_1' \equiv w_1\big)$;
2. $w \equiv w$ is the unit of the algebra B;
3. $\big(w_1 \equiv w_2\big) \wedge \big(w_2 \equiv w_3\big) \le \big(w_1 \equiv w_3\big)$;
4. For every n-ary operation $\omega \in \Omega$, where Ω is a signature of the variety Θ, we have

$$\Big((w_1 \equiv w_1') \wedge \ldots \wedge (w_n \equiv w_n')\Big) \le \Big(\omega(w_1, \ldots, w_n) \equiv \omega(w_1', \ldots, w_n')\Big).$$

Thus, the equality on the extended Boolean algebra B is symmetric, reflexive and transitive predicate which respects all operations on $W(X)$. We can write the identities controlling the interaction of morphisms with the constant of the form $w \equiv w'$, $w, w' \in W(X)$, as follows:

(a) $s_*(w \equiv w') = \big(s(w) \equiv s(w')\big)$;

(b) $\Big((s^x_w)_* a \wedge (w \equiv w')\Big) \le (s^x_{w'})_* a$, where $a \in \mathbb{H}$ and $s^x_w : W(X) \to W(X)$ is defined by $s^x_w(x) = w$ and $s^x_w(x') = x'$ for $x' \neq x$.

Item (a) is exactly condition (*4*) from Definition 2, item (b) follows from (*4*) and properties of the equality predicate "$\equiv$". In such a way we can add additional axioms to the item (*4*) in Definition 2, if we know specific properties of predicates on B corresponding to relation symbols from Ψ.

1.2.2.3 Halmos Algebras

It is known that categories and multi-sorted algebras are tightly connected. In this section we present the Halmos category as a multi-sorted Halmos algebra. For some reasons such a representation is very useful, for instance, in this case we can speak about congruences on algebras. Information about multi-sorted algebras can be found in [12, 16].

In Sect. 1.2.2.1 we have already defined the signature L_X of an extended Boolean algebra:

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\},$$

where M_X is a set of all $\varphi(w_1, \ldots, w_{n_\varphi})$, $\varphi \in \Psi$, $w_1, \ldots, w_{n_\varphi} \in W(X)$, $X \in \Gamma$. We extend L_X by the symbol s_* and denote the new signature by L_Θ. Thus,

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, M_X, s_*\}.$$

In the signature L_Θ the symbol s_* is reserved for multi-sorted operations of the type $(X; Y)$, where X, Y run through Γ. For the definition of a type of a multi-sorted operation see [12, 16].

Definition 3 We call an algebra $\mathbb{H} = (\mathbb{H}(X), X \in \Gamma)$ in the signature L_Θ a Halmos algebra, if

1. Every domain $\mathbb{H}(X)$ is an extended Boolean algebra in the signature L_X.
2. To every homomorphism of free algebras $s : W(X) \to W(Y)$ it corresponds the operation $s_* : \mathbb{H}(X) \to \mathbb{H}(Y)$ of the type $(X; Y)$ in such a way that

   2.1 the map $s_* : \mathbb{H}(X) \to \mathbb{H}(Y)$ is a homomorphism of the corresponding Boolean algebras;
   2.2 for a given $s^1 : W(X) \to W(Y)$, $s^2 : W(Y) \to W(Z)$ and $a \in \mathbb{H}(X)$ we have $s^2_*(s^1_*(a)) = (s^2 s^1)_*(a)$.

3. The identities controlling the interaction of the operations s_*, s^1_*, s^2_*: $\mathbb{H}(X) \to \mathbb{H}(Y)$ with quantifiers and constants repeat the ones from Definition 2, items (3) and (4).

1.2.2.4 Examples of Halmos Categories

The next two examples of Halmos categories are connected with the examples of extended Boolean algebras from Sect. 1.2.2.1.

Example 3 Category $Hal_\Theta(\mathscr{H})$. Objects of this category are extended Boolean algebras $Hal_\Theta(\mathscr{H})$ from Example 1 for various $X \in \Gamma$. Morphisms $s_* : Hal^X_\Theta(\mathscr{H}) \to Hal^Y_\Theta(\mathscr{H})$ are defined as follows: $s_*A = \{\mu : W(X) \to H \mid \mu s \in A\}$, where $A \subset Hom(W(X), H)$, $s : W(X) \to W(Y)$.

Remark 1 A homomorphism $s : W(X) \to W(Y)$ gives rise to a map $\widetilde{s} : Hom(W(Y), H) \to Hom(W(X), H)$ by the rule $\widetilde{s}(\mu) = \mu s$, which is a morphism in the category of affine spaces $\Theta^0(H)$ (see Sect. 1.2.1). If A is a subset of $Hom(W(X), H)$, then $s_* A$ is the full pre-image of A under $\widetilde{s}$.

Example 4 Category $\widetilde{\Phi}$. We define Halmos category $\widetilde{\Phi}$ in terms of multi-sorted algebras, which is more convenient. Consider once again the signature

$$L_\Theta = \{\vee, \wedge, \neg, \exists x, M_X, s_*\},\ x \in X,\ X \in \Gamma.$$

Denote by $M = (M_X, X \in \Gamma)$ the multi-sorted set with the domains M_X, where M_X is the set of all $\varphi(w_1, \dots, w_{n_\varphi})$, $\varphi \in \Psi$, $w_1, \dots, w_{n_\varphi} \in W(X)$. Each formula $\varphi(w_1, \dots, w_{n_\varphi}) \in M_X$ is a formula of the length zero and of the sort X. Let u be a formula of the length n and the sort X, and let $x \in X$. Then the formulas $\neg u$ and $\exists x u$ are the formulas of the same sort X and the length $(n + 1)$. Let now u_1 and u_2 be formulas of the same sort X and the length n_1 and n_2 accordingly. Then the formulas $u_1 \vee u_2$ and $u_1 \wedge u_2$ have the length $(n_1 + n_2 + 1)$ and the sort X. For the given $s : W(X) \to W(Y)$ the formula $s_* u$ is a formula of the length $(n + 1)$ and of the sort Y. By induction, one can define lengths and sorts of arbitrary formulas.

Let $\mathfrak{L}_X$ be the set of all formulas of the sort X constructed in such a way. Denote by

$$\mathfrak{L} = (\mathfrak{L}_X, X \in \Gamma)$$

the multi-sorted algebra in the signature L_Θ with domains $\mathfrak{L}_X$. By construction, the algebra $\mathfrak{L}$ is the absolutely free algebra of formulas over multi-sorted set $M = (M_X, X \in \Gamma)$. Define on $\mathfrak{L}$ the multi-sorted Lindenbaum–Tarski congruence $\tau = (\tau_X, X \in \Gamma)$ which works on components τ_X as follows:

$$u \tau_X v \text{ if and only if } \vdash (u \to v) \wedge (v \to u),$$

where $u, v \in \mathfrak{L}_X$. This means that two formulas u and v of the same sort are claimed to be equivalent if each of them is derivable from the other in the theory of multi-sorted Halmos algebras.

We define the Halmos algebra of formulas $\widetilde{\Phi}$ as a quotient algebra of the absolutely free algebra of formulas $\mathfrak{L}$ by the Lindenbaum–Tarski congruence τ, that is, $\widetilde{\Phi} = \mathfrak{L}/\tau$. It can be written as $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where $\Phi(X) = \mathfrak{L}_X/\tau_X$. By Definition 3 each component $\Phi(X)$ of multi-sorted Halmos algebra $\widetilde{\Phi}$ is an extended Boolean algebra of the sort X in the signature L_X.

Thus, simultaneously, we define the category $\widetilde{\Phi}$ with objects of the form $\Phi(X)$ and morphisms $s_* : \Phi(X) \to \Phi(Y)$, $X, Y \in \Gamma$. The next remark is connected with Remark 1.

Remark 2 Let a homomorphism $s : W(X) \to W(Y)$ be given. In parallel to the map $\widetilde{s} : Hom(W(Y), H) \to Hom(W(X), H)$, we define a map from the set of subsets in $\Phi(Y)$ to the set of subsets in $\Phi(X)$. We denote it by the same symbol $\widetilde{s}$ and define as $\widetilde{s}T = \{u \in \Phi(X) \mid s_* u \in T\}$, where $T \subset \Phi(Y)$. Then $\widetilde{s}T$ is the full pre-image of T under s_*.

Halmos categories $\widetilde{\Phi}$ and $Hal_\Theta(\mathscr{H})$ are tightly connected via homomorphism of extended Boolean algebras

$$Val^X_{\mathscr{H}} : \Phi(X) \to Bool(W(X), H).$$

Intuitively, the image of a formula $u \in \Phi(X)$ under the homomorphism $Val^X_{\mathscr{H}}$ is a value of u in the algebra H, i.e. $Val^X_{\mathscr{H}} u$ is the set of points in $Hom(W(X), H)$ satisfying u. For details see [17, 20].

1.2.3 The Galois Correspondence

In this section we introduce a correspondence between sets of formulas in the algebra $\Phi(X)$ and subsets of points from the affine space $Hom(W(X), H)$. In this concern we define the notion of a *logical kernel of a point*. This notion is closely related to the notion of a *type* from model theory [13]. Let μ be a point from the affine space $Hom(W(X), H)$.

Definition 4 The logical kernel $LKer(\mu)$ of a point μ is the set of all formulas $u \in \Phi(X)$ which hold true on the point μ, that is,

$$LKer(\mu) = \{u \in \Phi(X) \mid \mu \in Val^X_{\mathscr{H}}(u)\}$$

Remark 3 The logical kernel $LKer(\mu)$ of a point μ is a Boolean ultrafilter (maximal filter) in the algebra $\Phi(X)$ (see [18]).

Let T be a set of formulas from $\Phi(X)$. We define a set of points $T^L_{\mathscr{H}}$ in $Hom(W(X), H)$ as

$$T^L_{\mathscr{H}} = \{\mu : W(X) \to H \mid T \subset LKer(\mu)\}.$$

That is, $T^L_{\mathscr{H}}$ is a set of all points $\mu \in Hom(W(X), H)$ satisfying all formulas from $T \subset \Phi(X)$.

Take a set of points $A \subset Hom(W(X, H))$ and define a set of formulas $A^L_{\mathscr{H}}$ in $\Phi(X)$:

$$A^L_{\mathscr{H}} = \{u \in \Phi(X) \mid A \subset Val^X_{\mathscr{H}}(u)\}.$$

The set $A^L_{\mathscr{H}}$ is the set of all formulas $u \in \Phi(X)$ which hold true on all points from A.

The above defined correspondence between sets of formulas and sets of points is the Galois correspondence (see [11]). In the case of the Galois correspondence one can speak about Galois closures. In particular, subsets $T^L_{\mathscr{H}} \subset Hom(W(X), H)$ and $A^L_{\mathscr{H}} \subset \Phi(X)$ are Galois-closed.

We call the subset $T^L_{\mathscr{H}} \subset Hom(W(X), H)$ *definable set* presented by the set of formulas T. Note that the set $A^L_{\mathscr{H}}$ is a Boolean filter in the algebra $\Phi(X)$, it is called *$\mathscr{H}$-closed filter*. The constructed above Galois correspondence gives us a bijection between definable sets in $Hom(W(X), H)$ and $\mathscr{H}$-closed filters in the extended Boolean algebra $\Phi(X)$.

Proposition 1 *([18]) The intersection of $\mathscr{H}$-closed filters is an $\mathscr{H}$-closed filter.*

The next proposition describes one more property of $\mathscr{H}$-closed filters ([1, 17]).

Proposition 2 *Let a homomorphism of free algebras $s : W(X) \to W(Y)$ be given. If T is an $\mathscr{H}$-closed filter in $\Phi(Y)$, then $\widetilde{s}T$ is an $\mathscr{H}$-closed filter in $\Phi(X)$.*

The following proposition describes the relation between the Galois correspondence and morphisms in the categories $Hal_\Theta(\mathscr{H})$ and $\widetilde{\Phi}$ (see [1, 17]).

Proposition 3 *Let T be a set of formulas from $\Phi(X)$, A be a set of points in $Hom(W(X), H)$ and $s : W(X) \to W(Y)$ be a homomorphism of free algebras. Then*

1. $(s_*T)^L_{\mathscr{H}} = s_*T^L_{\mathscr{H}}$,
2. $s_*A^L_{\mathscr{H}} \subseteq (s_*A)^L_{\mathscr{H}}$.

Corollary 1 *If A is a definable set in $Hom(W(X), H)$ then s_*A is also a definable set.*

The similar relation takes place between definable sets, $\mathscr{H}$-closed filters and maps $\widetilde{s}$ (see [1, 17]).

Proposition 4 *Let T be a set of formulas from $\Phi(Y)$, A be a set of points in $Hom(W(Y), H)$. Let a homomorphism of free algebras $s : W(X) \to W(Y)$ be given. Then*

1. $\widetilde{s}(A^L_{\mathscr{H}}) = \left(\widetilde{s}A\right)^L_{\mathscr{H}}$,
2. $\widetilde{s}(T^L_{\mathscr{H}}) \subseteq \left(\widetilde{s}T\right)^L_{\mathscr{H}}$.

1.3 Knowledge Base Model

In this section we will introduce the concept of a *knowledge base model*.

1.3.1 What Is Knowledge?

Let us start with a discussion of what *knowledge* is. Speaking about knowledge we proceed from its representation in three components: *subject area of knowledge, description of knowledge, content of knowledge*.

Subject area of knowledge is presented by a model $\mathscr{H} = (H, \Psi, f)$, where H is an algebra in fixed variety of algebras Θ, Ψ is a set of relation symbols φ, f is an interpretation of each symbol φ in H.

Description of knowledge presents a syntactical component of knowledge. From algebraic viewpoint description of knowledge is a set of formulas T, more precise, it is an $\mathscr{H}$-filter in the algebra of formulas $\Phi(X)$, $X = \{x_1, \ldots, x_n\}$.

Content of knowledge is a subset in H^n, where H^n is the Cartesian power of H. Each content of a knowledge A corresponds to the description of a knowledge $T \subset \Phi(X)$, $|X| = n$. If we regard H^n as an affine space then this correspondence can be treated geometrically via Galois correspondence.

In order to describe the dynamic nature of a knowledge base two categories and a functor are introduced: *the category of knowledge description* $F_\Theta(\mathscr{H})$, *the category of knowledge content* $D_\Theta(\mathscr{H})$ and *the knowledge functor* $Ct_{\mathscr{H}}$.

These categories are defined using the machinery of *logical geometry* (see [2, 22]). Speaking about logic we presuppose that it has two important components:

$$syntax\ and\ semantics.$$

Applying techniques and notions of algebraic logic (see [9, 16], Sect. 1.2.2), we assign the appropriate algebraic structures to syntax and semantics: the category of knowledge description $F_\Theta(\mathscr{H})$ and the category of knowledge content $D_\Theta(\mathscr{H})$, correspondingly. Note that the paper contains a slightly different approach to the construction of the category of knowledge description $F_\Theta(\mathscr{H})$ and of the category of knowledge content $D_\Theta(\mathscr{H})$ than in [2, 22]. In particular, it gives us more appropriate connection between syntax and semantics, i.e., between the category of knowledge description $F_\Theta(\mathscr{H})$ and the category of knowledge content $D_\Theta(\mathscr{H})$. Namely, the new approach gives the dual isomorphism between $F_\Theta(\mathscr{H})$ and $D_\Theta(\mathscr{H})$ (see [1, 4] and Fig. 1.1).

Detailed description of categories $F_\Theta(\mathscr{H})$ and $D_\Theta(\mathscr{H})$ and the knowledge functor is given in [1]. Let us remind these constructions.

$$\begin{array}{lc} \textit{Syntax} & F_\Theta(\mathscr{H}) \\ & \Big\updownarrow {\scriptstyle Ct_{\mathscr{H}}} \\ \textit{Semantics} & D_\Theta(\mathscr{H}) \end{array}$$

Fig. 1.1 The knowledge functor $Ct_{\mathscr{H}}$: dual isomorphism

1.3.2 Category of Knowledge Description $F_\Theta(\mathscr{H})$

An object $F_\Theta^X(\mathscr{H})$ of the category $F_\Theta(\mathscr{H})$ is the lattice of all $\mathscr{H}$-closed filters in the algebra $\Phi(X)$, $X \in \Gamma$.

Remark 4 We cannot say that the usual set-theoretical union of $\mathscr{H}$-closed filters is an $\mathscr{H}$-closed filter. In order to obtain a lattice of $\mathscr{H}$-closed filters in $\Phi(X)$ we need a new operation

$$T_1 \overline{\cup} T_2 = (T_1 \cup T_2)^{LL}_{\mathscr{H}}.$$

Then all $\mathscr{H}$-closed filters in $\Phi(X)$ form a lattice with respect to operations $\overline{\cup}$ and $\cap$ (for details see [1, 17, 18]).

Let a homomorphism $s : W(X) \to W(Y)$ and $\mathscr{H}$-closed filters $T_1 \in \Phi(X)$ and $T_2 \in \Phi(Y)$ be given. We say that a map $[s_*] : T_1 \to T_2$ is *admissible*, if $s_* T_1 \subseteq T_2$. Remind that s_* is a map between $\mathscr{H}$-closed filters in $\Phi(X)$ and $\Phi(Y)$ induced by the corresponding morphism of the category $\widetilde{\Phi}$ (see Sect. 1.2.2).

A morphism between objects $F_\Theta^X(\mathscr{H})$ and $F_\Theta^Y(\mathscr{H})$ $[s_*] : F_\Theta^X(\mathscr{H}) \to F_\Theta^Y(\mathscr{H})$ is determined if $[s_*] : T_1 \to T_2$ is admissible for every $T_1 \in F_\Theta^X(\mathscr{H})$. We define a composition of morphisms $[s_*^1] : F_\Theta^X(\mathscr{H}) \to F_\Theta^Y(\mathscr{H})$ and $[s_*^2] : F_\Theta^Y(\mathscr{H}) \to F_\Theta^Z(\mathscr{H})$ as follows $[s_*^2] \circ [s_*^1] = [s_*^2 s_*^1]$.

1.3.3 Category of Knowledge Content $D_\Theta(\mathscr{H})$

An object $D_\Theta^X(\mathscr{H})$ of the category $D_\Theta(\mathscr{H})$ is the lattice of all definable sets in the affine space $Hom(W(X), H)$, $X \in \Gamma$.

Let a homomorphism $s : W(X) \to W(Y)$ and definable sets $A_2 \in Hom(W(Y), H)$ and $A_1 \in Hom(W(X), H)$ be given. We say that a map $[\widetilde{s}] : A_2 \to A_1$ is *admissible*, if $\widetilde{s} A_2 \subseteq A_1$. Remind that $\widetilde{s}$ is a map between definable sets in $Hom(W(Y), H)$ and $Hom(W(X), H)$ induced by the corresponding morphism of the category of affine spaces $\Theta^0(H)$ (see Sect. 1.2.1).

A morphism between objects $D_\Theta^Y(\mathscr{H})$ and $D_\Theta^X(\mathscr{H})$ $[\widetilde{s}] : D_\Theta^Y(\mathscr{H}) \to D_\Theta^X(\mathscr{H})$ is determined if $[\widetilde{s}] : A_2 \to A_1$ is admissible for every $A_2 \in D_\Theta^Y(\mathscr{H})$. We define a composition of morphisms $[\widetilde{s^1}] : F_\Theta^Z(\mathscr{H}) \to D_\Theta^Y(\mathscr{H})$ and $[\widetilde{s^2}] : D_\Theta^Y(\mathscr{H}) \to D_\Theta^X(\mathscr{H})$ as follows $[\widetilde{s^2}] \circ [\widetilde{s^1}] = [\widetilde{s^2}\widetilde{s^1}]$.

1.3.4 The Knowledge Functor $Ct_{\mathscr{H}}$

The category of knowledge description $F_\Theta(\mathscr{H})$ and the category of knowledge content $D_\Theta(\mathscr{H})$ are related by the knowledge functor:

$$Ct_{\mathscr{H}} : F_{\Theta}(\mathscr{H}) \to D_{\Theta}(\mathscr{H}),$$

which is defined on objects by $Ct_{\mathscr{H}}(F_{\Theta}^{X}(\mathscr{H})) = D_{\Theta}^{X}(\mathscr{H})$, and on morphisms by $Ct_{\mathscr{H}}([s_*]) = [\widetilde{s}]$, where $s : W(X) \to W(Y)$ is a given homomorphism of free algebras. Moreover, if $[s_*] : F_{\Theta}^{X}(\mathscr{H}) \to F_{\Theta}^{Y}(\mathscr{H})$ is a morphism in $F_{\Theta}(\mathscr{H})$, such that $[s_*] : T_1 \to T_2$, then $[\widetilde{s}] : D_{\Theta}^{Y}(\mathscr{H}) \to D_{\Theta}^{X}(\mathscr{H})$ is a morphism in $D_{\Theta}(\mathscr{H})$ defined by the rule $[\widetilde{s}] : (T_2)^{L}_{\mathscr{H}} \to (T_1)^{L}_{\mathscr{H}}$.

1.3.5 Definition of a Knowledge Base

Now we are on the point to give a definition of knowledge base model. Let a model $\mathscr{H} = (H, \Psi, f)$ be given.

Definition 5 A knowledge base $KB = KB(H, \Psi, f)$ is a triple $\big(F_{\Theta}(\mathscr{H}), D_{\Theta}(\mathscr{H}), Ct_{\mathscr{H}}\big)$, where $F_{\Theta}(\mathscr{H})$ is the category of knowledge description, $D_{\Theta}(\mathscr{H})$ is the category of knowledge content, and $Ct_{\mathscr{H}} : F_{\Theta}(\mathscr{H}) \to D_{\Theta}(\mathscr{H})$ is the contravariant functor.

Remark 5 We will use the term “a knowledge base” instead of a more precise “a knowledge base model”.

One can say that defined knowledge base model is a sort of automaton (see [16]), where queries are objects of the category of knowledge descriptions $F_{\Theta}(\mathscr{H})$, replies are objects of the category of knowledge content $D_{\Theta}(\mathscr{H})$. In order such an automaton to be viewed as a knowledge base, a connection with a particular data is supposed. This data is held in the subject area presented by a model $\mathscr{H} = (H, \Psi, f)$. The knowledge functor $Ct_{\mathscr{H}}$ gives a dynamical passage between queries and replies, namely, between categories $F_{\Theta}(\mathscr{H})$ and $D_{\Theta}(\mathscr{H})$. Moreover, this passage is a one-to-one correspondence.

Theorem 1 ([4]) *The knowledge functor $Ct_{\mathscr{H}}$ gives rise to contravariant isomorphism between the category of knowledge description $F_{\Theta}(\mathscr{H})$ and the category of knowledge content $D_{\Theta}(\mathscr{H})$.*

1.4 Knowledge Bases Equivalences

There are various approaches to specialize the meaning of knowledge bases equivalence. We are interested in a special kind of equivalence called *informational equivalence*.

1.4.1 Informationally Equivalent Knowledge Bases

Fix a variety of algebras Θ, algebras H_1 and H_2 from Θ and a set of relation symbols Ψ. Let two models $\mathscr{H}_1 = (H_1, \Psi, f_1)$ and $\mathscr{H}_2 = (H_2, \Psi, f_2)$ be given. Take the corresponding knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$.

Definition 6 Knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$ are called informationally equivalent, if the categories of knowledge description $F_\Theta(\mathscr{H}_1)$ and $F_\Theta(\mathscr{H}_2)$ are isomorphic.

In view of Theorem 1, the categories of knowledge description $F_\Theta(\mathscr{H}_1)$ and $F_\Theta(\mathscr{H}_2)$ are isomorphic if and only if the categories of knowledge content $D_\Theta(\mathscr{H}_1)$ and $D_\Theta(\mathscr{H}_2)$ are isomorphic. Thus, one can formulate Definition 6 in terms of isomorphism of the categories of knowledge content.

From now on our goal is to recognize the informational equivalence of knowledge bases in the sense of Definition 6. Our main interest is to find out how the informational equivalence is related to the logical description of knowledge bases. In this concern, there were defined *logically-geometrical equivalent (LG-equivalent), LG-isotypic and logically automorphically equivalent knowledge bases*. In this paper we concentrate our attention on LG-equivalent and LG-isotypic knowledge bases. Logically automorphically equivalent knowledge bases are considered in [3].

1.4.2 LG-equivalent and LG-isotypic Knowledge Bases

The notion of LG-equivalent (logically-geometrically equivalent) knowledge bases is based on geometrical approach, whereas LG-isotypic knowledge bases are defined using logical tools. But as we will see later these notions give us the same description of knowledge bases (see Theorem 3).

Both concepts, LG-equivalence and LG-isotypeness of knowledge bases, are defined via LG-equivalence and LG-isotypeness of subject areas of the given knowledge bases, that is, via LG-equivalence and LG-isotypeness of the corresponding models.

1.4.2.1 *LG*-equivalence and *LG*-isotypeness Of Models

Informally, we would like to define logically-geometrical equivalence of models $\mathscr{H}_1 = (H_1, \Psi, f_1)$ and $\mathscr{H}_2 = (H_2, \Psi, f_2)$ in such a way that the corresponding subject area algebras H_1 and H_2 have equal possibilities with respect to solution of logical formulas from any set of formulas T. This goal gives rise to the following definition.

Definition 7 The models $\mathscr{H}_1 = (H_1, \Psi, f_1)$ and $\mathscr{H}_2 = (H_2, \Psi, f_2)$ are called LG-equivalent if for any $X \in \Gamma$ and $T \subset \Phi(X)$ we have

$$T^{LL}_{\mathscr{H}_1} = T^{LL}_{\mathscr{H}_2}.$$

In other words, LG-equivalence means that any $\mathscr{H}_1$-closed filter in $\Phi(X)$ is $\mathscr{H}_2$-closed for every $X \in \Gamma$.

The concept of *LG-isotypic models* is defined using model-theoretical tools. Let a language $\mathbb{L}$ and a model $\mathscr{H}$ be given (for definitions see [13]).

Definition 8 A set T of consistent $\mathbb{L}$-sentences $u(x_1, \ldots, x_n)$ with free variables from $x_1, \ldots, x_n$ is called an n-type.

Definition 9 An n-type T is complete if for every $\mathbb{L}$-sentence $u(x_1, \ldots, x_n)$ with free variables from $x_1, \ldots, x_n$ either $u(x_1, \ldots, x_n)$ or $\neg u(x_1, \ldots, x_n)$ is in T.

Let $\overline{a} = (a_1, \ldots, a_n)$ be a point from H^n. Denote by $tp^{\mathscr{H}}(\overline{a})$ the set of all sentences $u(x_1, \ldots, x_n)$ in free variables $x_1, \ldots, x_n$ such that $\mathscr{H} \models u(a_1, \ldots, a_n)$. One can check that $tp^{\mathscr{H}}(\overline{a})$ is a complete n-type. We will also say that $tp^{\mathscr{H}}(\overline{a})$ is a *type of the point* $(a_1, \ldots, a_n) \in H^n$.

Definition 10 A complete n-type T is called realizable in $\mathscr{H}$ if there is a point $\overline{a} = (a_1, \ldots, a_n) \in H^n$ such that $T = tp^{\mathscr{H}}(\overline{a})$.

In this case we will say that T is a *type of the model* $\mathscr{H}$.

Remark 6 Later on we will be interested only in complete realizable n-types. We will call an n-type $tp^{\mathscr{H}}(\overline{a})$ from Definition 10 an X-MT-type (model-theoretical type), where $X = \{x_1, \ldots, x_n\}$.

Define now the concept of an LG-type (logically-geometrical type). Remind, that the set of all formulas in variables from X valid on $\mathscr{H}$ is called *the elementary X-theory* of $\mathscr{H}$ (see [13, 19]). Denote by $Th^X(\mathscr{H})$ the X-elementary theory of $\mathscr{H}$.

Definition 11 Every ultrafilter T in the algebra $\Phi(X)$ containing $Th^X(\mathscr{H})$ is called X-LG-type.

Definition 12 An ultrafilter T is called X-LG-type of the model $\mathscr{H}$ if there is a point $\mu : W(X) \to H$, such that $T = LKer(\mu)$.

In the latter case, we will say that the type T is *realizable in* $\mathscr{H}$ (cf. Definition 10). More details about connections between model-theoretical and logically-geometrical types can be found in [20]. We only mention the important result of Zhitomirski [23] which gives a bridge between MT- and LG-types. He showed that model-theoretical types of two points coincide if and only if their logically-geometrical types coincide. Thus, one can deal with MT- or LG-types depending on the needs.

Let $S^X_{\mathscr{H}}$ be a set of all realizable X-LG-types of a model $\mathscr{H}$, that is,

$$S^X_{\mathscr{H}} = \{LKer\mu \mid \mu \in Hom(W(X), H)\}.$$

Definition 13 The models $\mathscr{H}_1 = (H_1, \Psi, f_1)$ and $\mathscr{H}_2 = (H_2, \Psi, f_2)$ are called LG-isotypic if

$$S^X_{\mathscr{H}_1} = S^X_{\mathscr{H}_2}$$

for every $X \in \Gamma$.

The next theorem gives us a relationship between LG-equivalent and LG-isotypic models.

Theorem 2 *The models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-equivalent if and only if they are LG-isotypic.*

Proof Let models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) be LG-equivalent. This means that an ultrafilter $LKer\mu$ is $\mathscr{H}_1$-closed if and only if it is $\mathscr{H}_2$-closed for every $X \in \Gamma$ and $\mu \in Hom(W(X), H_1)$. Moreover, every $\mathscr{H}_2$-closed ulrtafilter is the logical kernel $LKer\nu$ for a certain point $\nu \in Hom(W(X), H_2)$. Thus, $LKer\mu = LKer\nu$ and $S^X_{\mathscr{H}_1} = S^X_{\mathscr{H}_2}$ for every $X \in \Gamma$.

Suppose that models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-isotypic. This means that for every $X \in \Gamma$ a subset $T \in \Phi(X)$ is an $\mathscr{H}_1$-closed ultrafilter if and only if it is an $\mathscr{H}_2$-closed ultrafilter. Since every $\mathscr{H}_1$-closed filter is an intersection of $\mathscr{H}_1$-closed ultrafilters, then it is an intersection of $\mathscr{H}_2$-closed ultrafilters as well. Remind that the intersection of $\mathscr{H}$-closed filters is an $\mathscr{H}$-closed filter (see Proposition 1). Thus, T is an $\mathscr{H}_1$-closed filter if and only if it is an $\mathscr{H}_2$-closed filter. This means that the models (H_1, Ψ, f_1) and (H_2, Ψ, f_2) are LG-equivalent.

The next section deals with LG-equivalent and LG-isotypic knowledge bases.

1.4.2.2 LG-equivalence and LG-isotypeness of Knowledge Bases

As we mentioned, LG-equivalence and LG-isotypeness for knowledge bases are defined via the similar notions for the models.

Let two models $\mathscr{H}_1 = (H_1, \Psi, f_1)$, $\mathscr{H}_2 = (H_2, \Psi, f_2)$ and the corresponding knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$ be given.

Definition 14 Knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$ are called LG-equivalent if the corresponding models $\mathscr{H}_1$ and $\mathscr{H}_2$ are LG-equivalent.

Definition 15 Knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$ are called LG-isotypic if the corresponding models $\mathscr{H}_1$ and $\mathscr{H}_2$ are LG-isotypic.

The next theorem is a straightforward corollary of Theorem 2.

Theorem 3 *Knowledge bases $KB(\mathscr{H}_1)$ and $KB(\mathscr{H}_2)$ are LG-equivalent if and only if they are LG-isotypic.* $\square$

1.4.3 LG-Equivalence and Informational Equivalence of Knowledge Bases

Now our main goal is to find out how the informational equivalence is related to the logically-geometrical description of knowledge bases. In this section we present results which demonstrate the relationship between LG-equivalent (LG-isotypic) and informationally equivalent knowledge bases.

We start from auxiliary results concerning subject areas of LG-equivalent knowledge bases. The next theorem gives the correspondence between definable sets over two LG-equivalent models that will be useful in subsequent considerations.

Theorem 4 *If the models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are LG-equivalent, then there is a one-to-one correspondence between X-definable sets over $\mathcal{H}_1$ and $\mathcal{H}_2$, for every $X \in \Gamma$.*

Proof Fix a set $X \in \Gamma$. Let $A \in Hom(W(X), H_1)$ be an X-definable set over $\mathcal{H}_1$. Then $\left(A^L_{\mathcal{H}_1}\right)^L_{\mathcal{H}_2}$ is an X-definable set over $\mathcal{H}_2$. We will show that the correspondence

$$\tau_X : A \to \left(A^L_{\mathcal{H}_1}\right)^L_{\mathcal{H}_2} \tag{1.1}$$

is a bijection.

Let us show that the map τ_X is injective. Let A and B be definable sets from $Hom(W(X), H_1)$. Suppose that

$$\left(A^L_{\mathcal{H}_1}\right)^L_{\mathcal{H}_2} = \left(B^L_{\mathcal{H}_1}\right)^L_{\mathcal{H}_2},$$

then $\left(A^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2}$ and $\left(B^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2}$ are $\mathcal{H}_2$-closed filters in $\Phi(X)$, and

$$\left(A^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2} = \left(B^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2}.$$

Since models $\mathcal{H}_1$ and $\mathcal{H}_2$ are LG-equivalent, then

$$\left(A^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2} = \left(A^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_1}, \quad \left(B^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_2} = \left(B^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_1}.$$

Hence,

$$\left(A^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_1} = \left(B^L_{\mathcal{H}_1}\right)^{LL}_{\mathcal{H}_1}, \quad A^{LL}_{\mathcal{H}_1} = B^{LL}_{\mathcal{H}_1}.$$

The sets A and B are definable, so $A = A^{LL}_{\mathcal{H}_1} = B^{LL}_{\mathcal{H}_1} = B$ and the map τ_X is injective.

If B is an X-definable set over $\mathcal{H}_2$ then the set $\left(B^L_{\mathcal{H}_2}\right)^L_{\mathcal{H}_1}$ is X-definable over $\mathcal{H}_1$. Thus, the correspondence between X-definable sets over $\mathcal{H}_1$ and $\mathcal{H}_2$ is surjective.

Let $s : W(X) \to W(Y)$ be a homomorphism of free algebras. Denote by $[\widetilde{s}]^0_{\mathcal{H}_i}$ a morphism from $D^Y_\Theta(\mathcal{H}_i)$ to $D^X_\Theta(\mathcal{H}_i)$ in the category $D_\Theta(\mathcal{H}_i)$, such that

$$[\widetilde{s}]^0_{\mathscr{H}_i} : A \to (\widetilde{s}A)^{LL}_{\mathscr{H}_i}$$

for every $A \in D^Y_\Theta(\mathscr{H}_i)$.

Remark 7 It is more correct to write $\widetilde{s}_{\mathscr{H}_i}A$ instead of $\widetilde{s}A$, but we will omit the subscribe $\mathscr{H}_i$ in this case.

The next two propositions give the connection between correspondence τ_X constructed in Theorem 4 and morphisms in categories $D_\Theta(\mathscr{H}_1)$ and $D_\Theta(\mathscr{H}_2)$ of lattices of definable sets over $\mathscr{H}_1$ and $\mathscr{H}_2$, correspondingly.

Proposition 5 *If models $\mathscr{H}_1 = (H_1, \Psi, f_1)$ and $\mathscr{H}_2 = (H_2, \Psi, f_2)$ are LG-equivalent, then the following commutative diagram takes place*

$$\begin{tikzcd}
A \arrow[r, "{[\widetilde{s}]^0_{\mathscr{H}_1}}"] \arrow[d, "\tau_Y"'] & (\widetilde{s}A)^{LL}_{\mathscr{H}_1} \arrow[d, "\tau_X"] \\
\left(A^L_{\mathscr{H}_1}\right)^L_{\mathscr{H}_2} \arrow[r, "{[\widetilde{s}]^0_{\mathscr{H}_2}}"] & \left((\widetilde{s}A)^L_{\mathscr{H}_1}\right)^L_{\mathscr{H}_2}
\end{tikzcd} \tag{1.2}$$

where A is a definable set in $Hom(W(Y), H_1)$, $[\widetilde{s}]^0_{\mathscr{H}_1}$ and $[\widetilde{s}]^0_{\mathscr{H}_2}$ are morphisms in categories $D_\Theta(\mathscr{H}_1)$ and $D_\Theta(\mathscr{H}_2)$ defined above.

Proof To prove the commutativity of this diagram we need to show that

$$[\widetilde{s}]^0_{\mathscr{H}_2}(\tau_Y(A)) = \left((\widetilde{s}A)^L_{\mathscr{H}_1}\right)^L_{\mathscr{H}_2}$$

and

$$\tau_X([\widetilde{s}]^0_{\mathscr{H}_1}(A)) = \left((\widetilde{s}A)^L_{\mathscr{H}_1}\right)^L_{\mathscr{H}_2}.$$

Indeed, by definitions of $[\widetilde{s}]^0_{\mathscr{H}_2}$ and τ_Y we have

$$[\widetilde{s}]^0_{\mathscr{H}_2}(\tau_Y(A)) = [\widetilde{s}]^0_{\mathscr{H}_2}\big((A^L_{\mathscr{H}_1})^L_{\mathscr{H}_2}\big) = \Big(\widetilde{s}\big((A^L_{\mathscr{H}_1})^L_{\mathscr{H}_2}\big)\Big)^{LL}_{\mathscr{H}_2}.$$

According to Proposition 4, $\left(\widetilde{s}A\right)^L_{\mathscr{H}} = \widetilde{s}(A^L_{\mathscr{H}})$, then $\Big(\widetilde{s}\big((A^L_{\mathscr{H}_1})^L_{\mathscr{H}_2}\big)\Big)^{LL}_{\mathscr{H}_2} = \Big(\widetilde{s}\big((A^L_{\mathscr{H}_1})^{LL}_{\mathscr{H}_2}\big)\Big)^L_{\mathscr{H}_2}$. Since models $\mathscr{H}_1$ and $\mathscr{H}_2$ are LG-equivalent, then $A^L_{\mathscr{H}_1}$ is an $\mathscr{H}_1$-closed and an $\mathscr{H}_2$-closed filter, simultaneously. This means that $(A^L_{\mathscr{H}_1})^{LL}_{\mathscr{H}_2} = A^L_{\mathscr{H}_1}$ and $\Big(\widetilde{s}\big((A^L_{\mathscr{H}_1})^{LL}_{\mathscr{H}_2}\big)\Big)^L_{\mathscr{H}_2} = \big(\widetilde{s}(A^L_{\mathscr{H}_1})\big)^L_{\mathscr{H}_2}$. Using Proposition 4, we have $\big(\widetilde{s}(A^L_{\mathscr{H}_1})\big)^L_{\mathscr{H}_2} = \big((\widetilde{s}A)^L_{\mathscr{H}_1}\big)^L_{\mathscr{H}_2}$. Thus, $[\widetilde{s}]^0_{\mathscr{H}_2}(\tau_Y(A)) = \big((\widetilde{s}A)^L_{\mathscr{H}_1}\big)^L_{\mathscr{H}_2}$.

Let us prove the equality $\tau_X([\widetilde{s}]^0_{\mathscr{H}_1}(A)) = \big((\widetilde{s}A)^L_{\mathscr{H}_1}\big)^L_{\mathscr{H}_2}$. By definitions of τ_X and $[\widetilde{s}]^0_{\mathscr{H}_1}$ we have $\tau_X([\widetilde{s}]^0_{\mathscr{H}_1}(A)) = \tau_X\big((\widetilde{s}A)^{LL}_{\mathscr{H}_1}\big) = \Big(\big((\widetilde{s}A)^{LL}_{\mathscr{H}_1}\big)^L_{\mathscr{H}_1}\Big)^L_{\mathscr{H}_2}$. Since $(\widetilde{s}A)^L_{\mathscr{H}_1}$ is