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# Multi-Valued Variational Inequalities and Inclusions

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# Multi-Valued Variational Inequalities and Inclusions

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*Dedicated to my wife Gudrun*

*S. Carl*

*Dedicated to my mother*

*V.K. Le*

# Preface

The foundation and a systematic study of variational inequalities date back to the 1960s of the last century and began with the work of Fichera [113] and Stampacchia [225, 226], which was motivated by problems in mechanics and potential theory, for example obstacle problems in elasticity, the Signorini problem [222], and the study of the capacity of sets. The rapid growth of the theory, made possible by the work of Lions and Stampacchia [187], Brézis [31, 32], Browder [35, 36], Kinderlehrer [145, 146], Duvaut and Lions [104], Friedman [114], Baiocchi and Capelo [16], Troianiello [234], Panagiotopoulos [203, 205], and many others, brought about important contributions to nonlinear and nonsmooth analysis, calculus of variations, optimization theory, and to several branches of mechanics, mathematical physics, and engineering (see e.g., [14, 77, 121, 133, 152, 212]).

Multi-valued variational inequalities have their origin in the study of (nonsmooth) locally Lipschitz continuous energy functionals under constraints, and arise as necessary conditions for critical points of such functionals, which can be expressed in the form of variational-hemivariational inequalities. This new type of variational inequalities, first introduced by Panagiotopoulos in the 1980s of the last century [203, 204], was closely related to the development of the by then new concept of Clarke's generalized gradient of locally Lipschitz functionals [79], and was used to model some mechanical problems governed by nonconvex, nonsmooth energy functionals, which naturally arise if nonmonotone, multi-valued constitutive laws are taken into account. Variational-hemivariational inequalities will be seen to be only particular cases of the multi-valued variational inequalities we are going to study in this monograph.

This book focuses on a large class of multi-valued variational differential inequalities and inclusions of nonpotential type of the form

$$u \in X \cap D(\partial\Psi) : 0 \in \mathcal{A}(u) + \mathcal{F}(u) + \partial\Psi(u) \quad \text{in } X^*, \tag{1}$$

whose leading differential operator $\mathcal{A} : X \to 2^{X^*}$ is a second-order Leray-Lions operator. The (multi-valued) lower order operator $\mathcal{F} : X \to 2^{X^*}$ in (1), which may depend on $u$ and its gradient $\nabla u$, is basically supposed to be only upper semicontinuous with respect to $u$ and $\nabla u$. The constraint is reflected by the subdifferential of a convex, lower semicontinuous, and proper functional $\Psi : X \to \mathbb{R} \cup \{+\infty\}$. In particular, constraints given by closed and convex subsets $K \subset X$, which amount to $\Psi = I_K$, where $I_K$ is the indicator function of $K$, are discussed in detail separately. Depending on the growth conditions imposed on the operators $\mathcal{A}$ and $\mathcal{F}$, we investigate (1) in different function spaces $X$ such as Sobolev spaces, Orlicz-Sobolev spaces, and Sobolev spaces with variable exponents. Besides the treatment of (1) in function spaces on bounded domains of $\mathbb{R}^N$, we also investigate (1) on unbounded domains, for which an appropriate setting is in Beppo-Levi spaces and weighted Lebesgue spaces. As is well known, the unboundedness of the domain under consideration causes a number of additional difficulties to the investigation, and therefore the analysis of (1) in this case requires new techniques and is not a straightforward extension of the study of its corresponding problem on bounded domains.

The main goal of this monograph is to provide a systematic, unified, and relatively self-contained exposition of existence and comparison principles of the multi-valued variational inequality (1) based on a suitably generalized method of sub-supersolution, which preserves the characteristic features of the commonly known sub-supersolution method for quasilinear elliptic problems. This method will be established not only for the stationary multi-valued variational inequality (1), but also for its evolutionary counterpart

$$u \in L^p(0,\tau;X) \cap D(\partial\Psi) : 0 \in u' + \mathcal{A}(u) + \mathcal{F}(u) + \partial\Psi(u) \quad \text{in } L^{p'}(0,\tau;X^*), \tag{2}$$

as well as for systems of (1) and (2), where $u' = \frac{du}{dt}$ denotes the generalized derivative in the sense of vector-valued distributions. It should be pointed out that in the treatment of the evolutionary multi-valued variational inequality (2), an additional difficulty arises, due to the presence of the indicator function $\Psi = I_K$, so that no growth condition can be assumed on $\partial\Psi$, and therefore, in general, no estimate of the time derivative $\frac{du}{dt}$ in the dual space, which would be necessary for proving existence of solutions, is available. In the case where $K$ has a nonempty interior, that is $\operatorname{int}(K) \neq \emptyset$, this difficulty can be easily overcome (see Section 5.3). However, the requirement that $\operatorname{int}(K) \neq \emptyset$ would exclude the important application to obstacle problems, since convex sets representing obstacles have, in general, empty interiors. This difficulty is overcome either by applying a penalty technique (see Chapter 5) or by requiring additional regularity assumptions on the operators $\mathcal{A}$ and $\mathcal{F}$ of (2) (see Chapter 7).

For the reader's orientation, we present in Chapter 1 some motivating examples and an outline of the topics studied in this monograph. In the treatment of the problems under consideration, a wide range of mathematical theories and methods from nonlinear and nonsmooth analysis, partial differential equations, and function spaces have been employed, a brief outline of which is provided in Chapter 2, in order to keep the volume mostly self-contained. The main materials form the contents of the next five chapters, from Chapter 3 to Chapter 7.

This treatise is an outgrowth of the authors' research on the subject during the past ten years. However, a great deal of the material presented here has been obtained only in recent years and appears for the first time in book form.

Our book is addressed to graduate students of mathematics and researchers in pure and applied mathematics, physics, and theoretical mechanics.

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Halle (Saale), Germany — Siegfried Carl
Rolla, MO, USA — Vy Khoi Le
February 2021

# Contents

# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | natural numbers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}_+$ | nonnegative real numbers |
| $\mathbb{R}^N$ | $N$-dimensional Euclidean space |
| $\Omega$ | open domain in $\mathbb{R}^N$ |
| $\partial\Omega$ | boundary of $\Omega$ |
| $\|E\|$ | Lebesgue measure of a subset $E \subset \mathbb{R}^N$ |
| a.a. | "almost all" with respect to the Lebesgue measure |
| a.e. | "almost every" with respect to the Lebesgue measure |
| iff | stands for "if and only if" |
| $X$ | real normed linear space |
| $X^*$ | dual space of $X$ |
| $\langle \cdot, \cdot \rangle$ | duality pairing |
| $X^{**}$ | bidual space of $X$ |
| $X_+$ | positive (or order) cone of $X$ |
| $X_+^*$ | dual order cone of $X$ |
| $x \wedge y$ | $\min\{x, y\}$ |
| $x \vee y$ | $\max\{x, y\}$ |
| $x^+$ | $\max\{x, 0\}$ |
| $x^-$ | $\max\{-x, 0\}$ |
| $\|x\|$ | absolute value of $x$ $(= x^+ + x^-)$ |
| $X \subset Y$ | $X$ is a subset of $Y$ including $X = Y$ |
| $2^X$ | power set of the set $X$, that is, the set of all subsets of $X$ |
| $\mathcal{K}(X)$ | set of all closed and convex subsets of $X$ |
| $\overline{K}$ | closure of a subset $K$ of $X$ |
| $\text{int}(K)$ | interior of $K$ |
| $X \hookrightarrow Y$ | $X$ is continuously embedded into $Y$ |
| $X \hookrightarrow\hookrightarrow Y$ | $X$ is compactly embedded into $Y$ |
| $\to$ | strong (norm) convergence |
| $\rightharpoonup$ | weak convergence |
| $\rightharpoonup^*$ | weak* convergence |

| | |
|---|---|
| $D(A)$ | domain of the operator $A$ |
| $\mathrm{Gr}(A)$ | graph of the mapping $A$ |
| $A^*$ | adjoint operator to $A$ |
| $\mathrm{dom}(J)$ | effective domain of the functional $J : X \to \mathbb{R} \cup \{+\infty\}$ |
| $J^*$ | conjugate convex functional of $J : X \to \mathbb{R} \cup \{+\infty\}$, $J^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - J(x)\}, x^* \in X^*$ |
| $I_K$ | indicator function, that is, $I_K(x) = 0$ if $x \in K$, $+\infty$ otherwise |
| $\chi_E$ | characteristic function of the set $E$ |
| $f'(u; h)$ | directional derivative |
| $D_G f$ | Gâteaux derivative |
| $D_F f$ or $f'$ | Fréchet derivative |
| $f^o(u; h)$ | generalized directional derivative |
| $\partial f$ | subdifferential of $f$ or Clarke's generalized gradient |
| $\nabla u$ | $(\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$, the gradient of $u$ |
| $\\|\nabla u\\|$ | stands for $\\| \, \lvert\nabla u\rvert \, \\|$ |
| $\Delta u$ | $\partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 + \cdots + \partial^2 u/\partial x_N^2$, the Laplacian of $u$ |
| $\Delta_p u$ | the $p$-Laplacian of $u$ |
| $C_c^\infty(\Omega)$ | space of infinitely differentiable functions with compact support in $\Omega$ |
| $\\|f\\|_{L^p(\Omega)}$ | $\left(\int_\Omega \lvert f\rvert^p dx\right)^{1/p}$, the $L^p$ norm |
| $L^p(\Omega)$ | space of $p$ integrable functions (whose $L^p$ norm is bounded) |
| $L^p_{\mathrm{loc}}(\Omega)$ | space of locally $p$ integrable functions |
| $L^q(\Omega, w)$ | weighted Lebesgue space with weight $w$ |
| $p'$ | Hölder conjugate to exponent $p$, $\frac{1}{p} + \frac{1}{p'} = 1$ |
| $p^*$ | critical Sobolev exponent, $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = \infty$ if $N \le p$ |
| $\\|f\\|_{W^{m,p}(\Omega)}$ | $\left(\sum_{\lvert\beta\rvert\le m} \int_\Omega \lvert D^\beta f\rvert^p dx\right)^{1/p}$, the Sobolev norm |
| $W^{m,p}(\Omega)$ | Sobolev space of functions with bounded $\\|\cdot\\|_{W^{m,p}(\Omega)}$-norm |
| $\gamma(u)$ or $\gamma u$ | trace of $u$ or generalized boundary values of $u$ |
| $W_0^{m,p}(\Omega)$ | completion of $C_c^\infty(\Omega)$ with respect to the norm $\\|\cdot\\|_{W^{m,p}(\Omega)}$ |
| $L_\Phi(\Omega)$ | Orlicz space |
| $\overline{\Phi}$ | Hölder conjugate of $N$-function $\Phi$, $\overline{\Phi}(t) = \sup_{s\in\mathbb{R}}\{ts - \Phi(s)\}$ |
| $\Phi^*$ | Sobolev conjugate of $N$-function $\Phi$ |
| $W^m L_\Phi(\Omega)$ | Orlicz-Sobolev space |
| $W_0^m L_\Phi(\Omega)$ | completion of $C_c^\infty(\Omega)$ with respect to the norm of $W^m L_\Phi(\Omega)$ |
| $L^{p(\cdot)}(\Omega)$ | Lebesgue space with variable exponent $p(\cdot)$ |
| $p'(\cdot)$ | Hölder conjugate of $p(\cdot)$, $p'(x) = \frac{p(x)}{p(x)-1}$ if $p(x) > 1$ |
| $p^*(\cdot)$ | Sobolev conjugate of $p(\cdot)$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, if $p(x) < N$, $p^*(x) = \infty$ if $p(x) \ge N$ |
| $p_-$ | $p_- = \min_{x\in\overline{\Omega}} p(x)$ for $p \in C(\overline{\Omega})$ |
| $p_+$ | $p_+ = \max_{x\in\overline{\Omega}} p(x)$ for $p \in C(\overline{\Omega})$ |
| $W^{m,p(\cdot)}(\Omega)$ | Sobolev space of order $m$ with variable exponent $p(\cdot)$ |
| $i^*_{q(\cdot)}$ | adjoint operator of the identity embedding |

| | |
|---|---|
| $i_{q(\cdot)}$ : | $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ |
| $D^{1,p}(\Omega)$ | Beppo-Levi space |
| $L^p(0, \tau; B)$ | space of $p$ integrable vector-valued functions $u : (0, \tau) \to B$ |
| $u'$ | also denoted by $\frac{du}{dt}$ or $u_t$ stands for the generalized derivative of the vector-valued function $u : (0, \tau) \to B$ |
| $C([0, \tau]; B)$ | space of continuous vector-valued functions $u : [0, \tau] \to B$ |
| $C^1([0, \tau]; B)$ | space of continuously differentiable vector-valued functions $u : [0, \tau] \to B$ |

# Chapter 1
# Introduction

In the study of a wide range of nonlinear elliptic and parabolic boundary value problems, the method of sub-supersolution has been proved to play an eminent role. This method is a powerful tool for establishing existence and enclosure results when coercivity of the operators related to the abstract formulation of the problems under consideration fails. Further qualitative properties such as the multiplicity and location of solutions or the existence of extremal solutions can also be investigated by means of the sub-supersolution method. As stationary and evolutionary variational inequalities of nonpotential type include, in general, nonlinear elliptic and parabolic boundary value problems as particular cases, it is desirable to extend the sub-supersolution method to variational inequalities in a way that preserves its characteristic features.

This book focuses on even more general multi-valued variational inequalities (MVIs in short) of possible nonpotential type of the abstract form

$$u \in X \cap D(\partial\Psi) : 0 \in \mathcal{A}(u) + \mathcal{F}(u) + \partial\Psi(u) \quad \text{in} \;\; X^*, \tag{1.1}$$

where $X$ stands for some function space with its dual space $X^*$, $\mathcal{A} : X \to 2^{X^*}$ is a second-order Leray-Lions operator, and $\mathcal{F} : X \to 2^{X^*}$ is a multi-valued lower order operator, which may depend on $u$ and its gradient $\nabla u$, and which, basically, is only supposed to be upper semicontinuous with respect to $u$ and $\nabla u$. The constraint imposed on the problem is reflected by the subdifferential of a convex, lower semicontinuous, and proper functional $\Psi : X \to \mathbb{R} \cup \{+\infty\}$, which includes, in particular, the important case of $\Psi = I_K$, the indicator function of some closed and convex set $K \subset X$.

An equivalent formulation of (1.1) reads as follows: Find $u \in X \cap D(\partial\Psi)$ such that there exist $\zeta^* \in \mathcal{A}(u)$ and $\eta^* \in \mathcal{F}(u)$ satisfying

$$\langle \zeta^* + \eta^*, v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall\, v \in X, \tag{1.2}$$



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where $D(\partial\Psi)$ denotes the domain of the subdifferential $\partial\Psi$ given by $D(\partial\Psi) = \{u \in X : \partial\Psi(u) \neq \emptyset\}$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X^*,\,X}$ denotes the duality pairing between $X^*$ and $X$. Besides the stationary MVI (1.1) we also study its evolutionary counterpart of the general form

$$u \in L^p(0,\tau;X) \cap D(\partial\Psi) : 0 \in u' + \mathcal{A}(u) + \mathcal{F}(u) + \partial\Psi(u) \quad \text{in } L^{p'}(0,\tau;X^*), \tag{1.3}$$

where $\mathcal{A}, \mathcal{F} : L^p(0,\tau;X) \to 2^{L^{p'}(0,\tau;X^*)}$ and $\partial\Psi : L^p(0,\tau;X) \cap D(\partial\Psi) \to 2^{L^{p'}(0,\tau;X^*)}$ are the corresponding time-dependent multi-valued operators, and $u' = \frac{du}{dt}$ denotes the generalized derivative in the sense of vector-valued distributions.

Our main goal is to present a systematic, unified, and self-contained exposition of existence and enclosure results based on a suitably generalized method of sub-supersolution, which is especially useful when the coercivity of the operators in (1.1) or (1.3) is not available. Let us present now two elementary motivating examples for stationary MVIs. Multi-valued variational inequalities have their background in the variational study of critical levels of certain (nonsmooth) locally Lipschitz energy functionals under constraints, and arise as necessary conditions for critical points of such functionals, such as local extrema. Consider first the following simple example on the real line.

*Example 1.1* Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, and $K = [a,b]$ be a closed interval of $\mathbb{R}$. Assume that the restriction $\Phi|_K$ attains its minimum at $u \in K$, that is,

$$\Phi(u) = \inf_{v\in K} \Phi(v). \tag{1.4}$$

This minimization problem on $K$ is equivalent to

$$u \in K : \inf_{v\in\mathbb{R}} [\Phi(v) + I_K(v)] = \Phi(u) + I_K(u) = \Phi(u), \tag{1.5}$$

where $I_K : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of $K$ defined by

$$I_K(v) = \begin{cases} 0 & \text{if} \quad v \in K, \\ +\infty & \text{if} \quad v \in X \setminus K. \end{cases}$$

By the convexity of $I_K$, the function $I = \Phi + I_K$ satisfies the following inequality for any $t \in (0,1)$ and $v \in \mathbb{R}$

$$\begin{aligned} 0 &\le I(u + t(v-u)) - I(u) = \Phi(u + t(v-u)) - \Phi(u) + I_K(u + t(v-u)) - I_K(u) \\ &\le \Phi(u + t(v-u)) - \Phi(u) + t(I_K(v) - I_K(u)). \end{aligned}$$

Dividing the last inequality by $t > 0$ and passing to the $\limsup_{t\to 0+}$, we obtain the following necessary condition for the minimum point $u$:

$$0 \le \Phi^o(u; v-u) + I_K(v) - I_K(u), \quad \forall\, v \in \mathbb{R}, \tag{1.6}$$

where $\Phi^o(u; \varrho)$ denotes Clarke's generalized directional derivative of $\Phi$ at $u$ in the direction $\varrho$ (see Definition 2.54 in Chapter 2). Inequality (1.6) may be considered as a variational-hemivariational inequality on the real line. In deriving (1.6) we have employed the relation

$$\limsup_{t\to 0+} \frac{\Phi(u + t(v-u)) - \Phi(u)}{t} \le \Phi^o(u; v-u),$$

which readily follows from the definition of Clarke's generalized directional derivative. Now let us verify that inequality (1.6) is equivalent to the inclusion:

$$u \in D(\partial I_K) = K : 0 \in \partial\Phi(u) + \partial I_K(u), \tag{1.7}$$

where $\partial\Phi(u)$ stands for Clarke's generalized gradient of $\Phi$ at $u$, and $\partial I_K(u)$ stands for the subdifferential of the convex function $I_K$ in the sense of convex analysis (see Definition 2.52 in Chapter 2). To show the equivalence of (1.6) and (1.7), assume $u$ satisfies (1.6), which yields by setting $v - u = w$,

$$0 \le \Phi^o(u; w) + I_K(u+w) - I_K(u), \quad \forall\, w \in \mathbb{R}. \tag{1.8}$$

Since $\Phi^o(u; 0) = 0$, we see from (1.8) that 0 is the (global) minimum of the convex function $w \mapsto \Phi^o(u; w) + I_K(u+w) - I_K(u)$, which implies that $u \in \mathrm{dom}(I_K)$ and

$$0 \in \partial(\Phi^o(u; \cdot) + I_K(u+\cdot) - I_K(u))(0) = \partial\Phi^o(u; \cdot)(0) + \partial I_K(u+\cdot)(0).$$

From the definition of Clarke's gradient (see Definition 2.56), we see that $\partial\Phi^o(u; \cdot)(0) = \partial\Phi(u)$, and therefore,

$$0 \in \partial\Phi(u) + \partial I_K(u),$$

which is (1.7). Conversely, let $u$ satisfy (1.7). Then there is an $\eta \in \partial\Phi(u)$ and a $\xi \in \partial I_K(u)$ such that $\eta + \xi = 0$. Hence, by using the definition of Clarke's gradient $\partial\Phi$ and the subdifferential $\partial I_K(u)$ we have

$$0 = (\eta+\xi)(v-u) = \eta\,(v-u) + \xi\,(v-u) \le \Phi^o(u; v-u) + I_K(v) - I_K(u), \ \forall\, v \in \mathbb{R},$$

which is (1.6). We note that the inclusion (1.7) can be rewritten in the form $-\eta \in \partial I_K(u)$ with $\eta \in \partial\Phi(u)$, which amounts to

$$I_K(v) \geq I_K(u) + (-\eta)(v-u), \quad \forall\, v \in \mathbb{R},$$

or equivalently,

$$u \in K,\ \exists\, \eta \in \partial\Phi(u) : \eta\,(v-u) \geq 0, \quad \forall\, v \in K, \tag{1.9}$$

which may be seen as a multi-valued variational inequality on the real line. In the particular case where $\Phi : \mathbb{R} \to \mathbb{R}$ is smooth, that is, $\Phi \in C^1(\mathbb{R})$, we have $\partial\Phi(u) = \{\Phi'(u)\}$, and thus the multi-valued variational inequality (1.9) reduces to the following single-valued variational inequality

$$u \in K : \Phi'(u)\,(v-u) \geq 0, \quad \forall\, v \in K.$$

*Example 1.2* Let $X = W_0^{1,2}(\Omega)$ be the usual first-order Sobolev space with standard notation (cf. Section 2.1.2) and consider the energy functional $E$ on $X$ given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega j(u)\,dx, \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, and $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, whose Clarke's generalized gradient $\partial j$ satisfies the following growth condition:

$$\sup\{|\eta| : \eta \in \partial j(s)\} \leq c(1+|s|), \quad \forall\, s \in \mathbb{R}. \tag{1.11}$$

Due to the growth condition (1.11) one readily verifies that $E : X \to \mathbb{R}$ is well defined. Set

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx, \text{ and } J(u) = \int_\Omega j(u)\,dx.$$

Clearly, $\Phi \in C^1(X)$, and by Aubin-Clarke's Theorem (see Theorem 2.61 in Chapter 2) the functional $J : L^2(\Omega) \to \mathbb{R}$ is Lipschitz continuous on bounded sets of $L^2(\Omega)$ and its Clarke's generalized gradient satisfies

$$\partial J(u) \subset \{v \in L^2(\Omega) : v(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega\}, \tag{1.12}$$

and if $j : \mathbb{R} \to \mathbb{R}$ is regular in the sense of Definition 2.55, then $J$ is regular and equality holds in (1.12). Let $i : X \to L^2(\Omega)$ denote the embedding operator, which is known to be compact. Then $J \circ i : X \to \mathbb{R}$ is locally Lipschitz, and thus $E = \Phi + J \circ i : X \to \mathbb{R}$, being the sum of a differentiable functional and a locally Lipschitz functional, is also locally Lipschitz.

Let $K \subset X$ be closed and convex and $I_K$ its corresponding indicator function. As in Example 1.1 our goal is to derive a necessary condition for the minimum point of the nonsmooth (locally Lipschitz) energy functional $E$ under the constraint $K$. Assume $u \in K$ satisfies

$$u \in K : E(u) = \inf_{v \in K} E(v) = \inf_{v \in X} [E(v) + I_K(v)]. \tag{1.13}$$

Repeating the arguments in Example 1.1 leads to the following variational-hemivariational inequality, as a necessary condition for a minimizer $u$,

$$0 \le E^o(u; v - u) + I_K(v) - I_K(u), \quad \forall\, v \in X = W_0^{1,2}(\Omega), \tag{1.14}$$

which is equivalent to the inclusion

$$u \in D(\partial I_K) = K : 0 \in \partial E(u) + \partial I_K(u), \tag{1.15}$$

where $\partial E(u)$ stands for Clarke's gradient of the locally Lipschitz functional $E$ at $u$, and $\partial I_K(u)$ is the subdifferential of the (convex) indicator function $I_K$ at $u$. By applying Proposition 2.26 of Chapter 2 to $E = \Phi + J \circ i$, and taking into account that $\Phi \in C^1(X)$, we obtain

$$\partial E(u) = \Phi'(u) + \partial(J \circ i)(u).$$

The chain rule for Clarke's gradient (see Corollary 2.16) yields

$$\partial(J \circ i)(u) = i^* \partial J(iu),$$

where $i^* : L^2(\Omega) \to X^*$ denotes the adjoint operator of the embedding $i : X \to L^2(\Omega)$ given by

$$\langle i^* \eta, u \rangle = (\eta, iu) = \int_\Omega \eta\, u\, dx, \quad \eta \in L^2(\Omega),\ u \in X.$$

Thus we arrive at the formula

$$\partial E(u) = \Phi'(u) + i^* \partial J(iu), \quad \forall\, u \in X. \tag{1.16}$$

Finally, from (1.15) and (1.16) we obtain the following necessary condition for a minimum point of $E$:

$$u \in D(\partial I_K) = K :\ 0 \in \Phi'(u) + i^* \partial J(iu) + \partial I_K(u) \ \text{ in } \ X^*, \tag{1.17}$$

which means that there is an $\eta \in \partial J(iu) = \partial J(u)$ and a $\xi \in \partial I_K(u)$ such that

$$0 = \Phi'(u) + i^*\eta + \xi,$$

or, equivalently,

$$-(\Phi'(u) + i^*\eta) = \xi \in \partial I_K(u),$$

which yields the necessary condition

$$u \in K,\ \exists\, \eta \in \partial J(u):\ \langle \Phi'(u) + i^*\eta, v - u\rangle \geq 0, \quad \forall\, v \in K. \tag{1.18}$$

As is well known we have

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \nabla v\, dx = \langle -\Delta u, v\rangle, \quad \forall\, v \in X.$$

Denoting by $F$ the Nemytskij operator associated with the multi-valued function $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, that is, $F(u)(x) = \partial j(u(x))$ and taking (1.12) into account, we arrive at the following necessary condition for minimum points of $E$ under the constraint $K$:

$$u \in D(\partial I_K) \cap X:\ \ 0 \in -\Delta u + \mathcal{F}(u) + \partial I_K(u) \ \text{ in } \ X^*, \tag{1.19}$$

where $\mathcal{F} = i^* \circ \partial j \circ i$ is the multi-valued operator generated by $\partial j$. The problem (1.19) is thus a particular case of the abstract MVI (1.1) with $\mathcal{A} = -\Delta$, $\mathcal{F} = i^* \circ \partial j \circ i$, and $\Psi = I_K$.

As the two examples above suggest, MVIs originate from the study of critical levels of nonsmooth energy functionals, so-called superpotentials. However, important applications in mechanics (see, e.g., [202–204, 224]) are described by variational-hemivariational inequalities or differential inclusions that do not originate from superpotentials. Those applications motivate our study of MVIs of the form (1.1) and (1.3) that do not necessarily have variational structure, and of which variational-hemivariational inequalities are particular cases only. However, rather than dealing with specific applications and modeling, in this treatise we concentrate primarily on mathematical theories for MVIs.

We would like to point out that the main goal of this monograph is to establish a unified method of sub-supersolution for the general MVIs (1.1) and (1.3) that will allow us to prove not only existence and enclosure results, but also to prove certain qualitative properties of their solution sets. We also note that by specifying the multi-valued operators $\mathcal{A}$, $\mathcal{F}$ and the functional $\Psi$, the MVIs considered here contain a wide variety of boundary value problems of quasilinear elliptic and parabolic inclusions, inequalities, and equations as particular cases.

This book is essentially an outgrowth of the authors' research on the subject during the past ten years. It consists of seven chapters including this introductory chapter.

Chapter 2 is of auxiliary nature and provides needed mathematical prerequisites to make the book relatively self-contained, such as, major function spaces, main results about abstract nonlinear operator equations, and basic concepts of functional analysis and nonsmooth analysis.

Chapter 3 deals with coercive as well as noncoercive single- and multi-valued elliptic and parabolic equations and inclusions governed by general (nonpotential type) Leray-Lions operators and (nonpotential type) single- and multi-valued lower order terms of which Clarke's generalized gradient is only a special case. The approach to study noncoercive problems is based on the sub-supersolution method established in this chapter, which may therefore be considered a preparatory chapter for introducing the methods and techniques that will be used in later chapters in more advanced and generalized settings.

Chapter 4 deals with general multi-valued elliptic variational inequalities of the form (1.1), where the constraints are given by closed convex sets, that is, where $\Psi = I_K$ with $K$ being a closed and convex set. Here the sub-supersolution method is developed in its full generality to treat problems that lack coercivity and to prove the existence of extremal solutions which requires the elaboration of new and subtle techniques. The sub-supersolution method established here allows to verify the equivalence between generalized variational-hemivariational inequalities and a particular class of multi-valued elliptic variational inequalities. Further, MVIs with discontinuously perturbed lower order multi-valued terms are investigated. Finally, the sub-supersolution method is extended to systems of MVIs in Orlicz-Sobolev spaces.

Chapter 5 is devoted to multi-valued evolutionary variational inequalities of the form (1.3) and related systems under constraints given by closed convex sets, that is, $\Psi = I_K$. It should be noted that unlike in the stationary case (1.1), in the treatment of its evolutionary counterpart an additional difficulty arises. This difficulty is due to the appearance of $\Psi = I_K$, so that no growth condition can be assumed on $\partial I_K$, and therefore, in general, no estimate of the time derivative $du/dt$ of the Banach-valued function $t \mapsto u(t)$ in the dual space is available, which would be needed for proving existence of solutions. In the case where $K$ has a nonempty interior, that is $\text{int}\,(K) \neq \emptyset$, this difficulty can be easily overcome, since such an assumption typically allows the application of Rockafellar's theorem about sums of maximal monotone operators, which facilitates the study of parabolic variational inequalities considerably by the implementation of arguments and results for elliptic variational inequalities to parabolic variational inequalities (see Section 5.3). However, the condition $\text{int}\,(K) \neq \emptyset$ would exclude the investigation of certain most important classes of evolutionary variational inequalities such as parabolic obstacles problems, in which the associated closed and convex set $K$ representing the obstacle has an empty interior, that is, $\text{int}\,(K) = \emptyset$. In this chapter, we deal with this difficulty by using a penalty technique, which allows us to treat general obstacle problems.

In Chapter 6 we investigate MVIs (1.1) and (1.3) with $\Psi = I_K$ in unbounded domains, which-as is well known-causes a number of additional difficulties, and therefore cannot be considered as just a straightforward extension of the bounded domain problems. Beppo-Levi spaces and weighted Lebesgue spaces play an important role to overcome a number difficulties that arise in the functional analytic treatment of such problems in unbounded domains, and new techniques such as the Kelvin transform are developed to treat problems in exterior domains.

Finally, in Chapter 7 the sub-supersolution method is extended to MVIs (1.1) and (1.3) with general convex, lower semicontinuous, and proper functionals $\Psi$. The convex functionals are seen here as characterizations of various constraints imposed on the problems, as well as potential functionals of possibly multi-valued leading operators. Compared to the case of MVIs on closed and convex sets, this more general situation is not a direct extension and requires the introduction of new concepts and implementation of new techniques in both classes of stationary and evolutionary MVIs. We also investigate in this chapter stationary MVIs, formulated in Sobolev spaces with variable exponents, in which the lower order terms may depend on both the unknown function $u$ and its gradient $\nabla u$.

# Chapter 2
# Mathematical Preliminaries

In this chapter we provide definitions and theorems that will be used in the sequel pertaining to major function spaces, theory of abstract nonlinear operator equations governed by (multi-valued) pseudomonotone operators and their evolutionary counterparts, as well as nonsmooth analysis. Most of these results can be found in textbooks or monographs, and are given without proof or short sketch of proof only.

## 2.1 Function Spaces

### *2.1.1 Operators in Normed Linear Spaces*

The purpose of this section is to provide a survey of basic results from functional analysis that will be used in the sequel. However, we will assume that the reader is familiar with some elementary notions such as, for example, metric spaces, Banach spaces, and Hilbert spaces, as well as notions related to the topological structure of these spaces. Unless otherwise indicated, all linear spaces considered in this book are assumed to be defined over the real number field $\mathbb{R}$. The proofs of the results presented in this section can be found in standard textbooks, for example, [2, 17, 33, 153, 214, 242].
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces, and let

$$A : D(A) \subset X \to Y$$

be an operator with domain $D(A)$ and range denoted by range$(A)$. In case that $D(A) = X$ we write

$$A : X \to Y.$$

 
S. Carl, V. K. Le, *Multi-Valued Variational Inequalities and Inclusions*,
Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-65165-7_2

Note, usually we drop the subscripts $X$ and $Y$ in the notation of the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, in case there is no ambiguity. By $u_n \to u$ in $X$ we denote the norm-convergence, and by $u_n \rightharpoonup u$ we denote the weak-convergence of a sequence $(u_n) \subset X$.

**Definition 2.1** Let $A : D(A) \subset X \to Y$.

(i) $A$ is continuous at the point $u \in D(A)$ iff for each sequence $(u_n)$ in $D(A)$,

$$u_n \to u \quad \text{implies} \quad Au_n \to Au.$$

The operator $A : D(A) \subset X \to Y$ is called continuous iff it is continuous at each point $u \in D(A)$.

(ii) $A$ is called compact iff $A$ is continuous, and $A$ maps bounded sets into relatively compact sets.

(iii) $A$ is called completely continuous iff for every sequence $(u_n)$ in $D(A)$ with $u_n \rightharpoonup u$ with $u \in D(A) \subset X$, it follows $Au_n \to Au$ in $Y$. This means, a completely continuous operator is sequentially continuous from $D(A)$ with the relative weak topology into $Y$ with the strong (norm) topology.

An immediate consequence from the definitions above along with the Eberlein-Smulian Theorem 2.7 about reflexive Banach spaces yields the following corollary.

**Corollary 2.1** *If $X$ is a reflexive Banach space, $D(A) \subset X$ is nonempty, closed, and convex, and $A : D(A) \subset X \to Y$ is completely continuous, then $A : D(A) \subset X \to Y$ is compact.*

For compact operators the following fixed-point theorem due to Schauder holds.

**Theorem 2.1 (Schauder's Fixed-Point Theorem)**
*Let $X$ be a Banach space, and let*

$$A : M \to M$$

*be a compact operator that maps a nonempty subset $M$ of $X$ into itself. Then $A$ has a fixed point provided $M$ is bounded, closed, and convex.*

In finite-dimensional normed linear spaces Theorem 2.1 reduces to Brouwer's fixed-point theorem.

**Corollary 2.2 (Brouwer's Fixed-Point Theorem)**
*If the operator*

$$A : M \to M$$

*is continuous, then $A$ has a fixed point provided $M$ is a compact, convex, nonempty subset in a finite-dimensional normed linear space.*

Let

$$A : D(A) \subset X \to Y$$

be a *linear* operator, which means that the domain $D(A)$ of the operator $A$ is a linear subspace of $X$ and $A$ satisfies

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{for all } u,\ v \in D(A),\ \alpha,\ \beta \in \mathbb{R}.$$

**Proposition 2.1** *Let $A : X \to Y$ be a linear operator. Then the following two conditions are equivalent.*

*(i) $A$ is continuous.*
*(ii) $A$ is bounded, that is, there is a constant $c > 0$ such that*

$$\|Au\| \le c\|u\| \quad \textit{for all } \ u \in X.$$

For a linear continuous operator $A : X \to Y$, the operator norm $\|A\|$ is defined by

$$\|A\| = \sup_{\|u\| \le 1} \|Au\|,$$

which can easily be shown to be equal to

$$\|A\| = \sup_{\|u\| = 1} \|Au\|.$$

**Proposition 2.2** *Let $L(X, Y)$ denote the space of linear continuous operators $A : X \to Y$, where $X$ is a normed linear space and $Y$ is a Banach space. Then $L(X, Y)$ is a Banach space with respect to the operator norm.*

**Definition 2.2** Let

$$A : D(A) \subset X \to Y$$

be a linear operator. The graph of $A$ denoted by $\mathrm{Gr}(A)$ is defined by the subset

$$\mathrm{Gr}(A) = \{(u, Au) : u \in D(A)\}$$

of the product space $X \times Y$. The operator $A$ is called closed (or graph-closed) iff $\mathrm{Gr}(A)$ is closed in $X \times Y$, which means that for each sequence $(u_n)$ in $D(A)$ it follows from

$$u_n \to u \ \text{ in } X \quad \text{and} \quad Au_n \to v \ \text{ in } Y,$$

that $u \in D(A)$ and $v = Au$. Finally, on $D(A)$ the so-called graph norm $\|\cdot\|_A$ is defined by

$$\|u\|_A = \|u\| + \|Au\| \quad \text{for } u \in D(A).$$

**Corollary 2.3** *If $X$ and $Y$ are Banach spaces and $A : D(A) \subset X \to Y$ is closed, then $D(A)$ equipped with the graph norm, that is, $(D(A), \|\cdot\|_A)$ is a Banach space.*

**Theorem 2.2 (Banach's Closed Graph Theorem)**
*Let $X$ and $Y$ be Banach spaces. Then, each closed linear operator $A : X \to Y$ is continuous.*

For completeness we shall recall the Uniform Boundedness Theorem and the Open Mapping Theorem, which together with Banach's Closed Graph Theorem are all consequences of Baire's Theorem.

**Theorem 2.3 (Uniform Boundedness Theorem)**
*Let $\mathcal{F}$ be a nonempty set of continuous maps*

$$F : X \to Y,$$

*where $X$ is a Banach space and $Y$ is a normed linear space. Assume that*

$$\sup_{F\in\mathcal{F}} \|Fu\| < \infty \quad \textit{for each } u \in X.$$

*Then there exists a closed ball $\overline{B}$ in $X$ of positive radius such that*

$$\sup_{u\in\overline{B}}(\sup_{F\in\mathcal{F}} \|Fu\|) < \infty.$$

**Corollary 2.4 (Banach-Steinhaus Theorem)**
*Let $\mathcal{L} \subset L(X, Y)$ be a nonempty set of linear continuous operators*

$$A : X \to Y,$$

*where $X$ is a Banach space and $Y$ is a normed linear space. Assume that*

$$\sup_{A\in\mathcal{L}} \|Au\| < \infty \quad \textit{for each } u \in X.$$

*Then* $\sup_{A\in\mathcal{L}} \|A\| < \infty$.

**Theorem 2.4 (Banach's Open Mapping Theorem)**
*Let $X$ and $Y$ be Banach spaces, and $A : X \to Y$ be a linear continuous operator. Then the following two conditions are equivalent.*

*(i) A is surjective.*
*(ii) A is open, which means that A maps open sets onto open sets.*

**Corollary 2.5 (Banach's Continuous Inverse Theorem)**
*Let $X$ and $Y$ be Banach spaces, and $A : X \to Y$ be a linear continuous operator. If the inverse operator*

$$A^{-1} : Y \to X$$

*exists, then $A^{-1}$ is continuous.*

**Definition 2.3 (Embedding Operator)**
Let $X$ and $Y$ be normed linear spaces with

$$X \subset Y.$$

The embedding operator $i : X \to Y$ is defined by $i(u) = u$, that is, $i$ is the identity operator from $X$ into $Y$.

(i) The embedding $X \subset Y$ is called continuous and denoted by $X \hookrightarrow Y$ iff the embedding operator $i : X \to Y$ is continuous, that is, there exists a constant $c > 0$ such that

$$\|u\|_Y \leq c\,\|u\|_X \quad \text{for all} \;\; u \in X.$$

(ii) The embedding $X \subset Y$ is called compact and denoted by $X \hookrightarrow\hookrightarrow Y$ iff the embedding operator $i : X \to Y$ is compact, that is, $i$ is continuous and each bounded sequence $(u_n)$ in $X$ has a subsequence that converges in $Y$.

*Remark 2.1* More generally, one can define a continuous embedding of a normed linear space $X$ into a normed linear space $Y$, whenever there exists a linear, continuous, and injective operator $i : X \to Y$. Similarly, $X$ is compactly embedded into $Y$ iff there exists a linear, compact, and injective operator $i : X \to Y$.

**Duality in Banach Spaces**

**Definition 2.4** Let $X$ be a normed linear space. A linear continuous functional on $X$ is a linear continuous operator

$$f : X \to \mathbb{R}.$$

The set of all linear continuous functionals on $X$ is called the dual space $X^*$ of $X$, that is, $X^* = L(X, \mathbb{R})$. For the image $f(u)$ of the functional $f$ at $u \in X$ we write

$$\langle f, u \rangle = f(u) \quad u \in X, \;\; f \in X^*,$$

and $\langle \cdot, \cdot \rangle$ is called the duality pairing.

According to the definition of the operator norm, the norm of $f$ is given through

$$\|f\| = \sup_{\|u\|\le 1} |\langle f, u\rangle|.$$

As a consequence of Proposition 2.2 we get the following result.

**Corollary 2.6** *Let $X$ be a normed linear space. Then the dual space $X^*$ is a Banach space with respect to the norm $\|f\|$ for $f \in X^*$.*

The most important theorem about the structure of linear functionals on normed linear spaces is the Hahn-Banach Theorem.

**Theorem 2.5 (Hahn-Banach Theorem)**
*Let $X$ be a normed linear space. Assume $M$ is a linear subspace of $X$, and $F : M \to \mathbb{R}$ is a linear functional such that*

$$|F(u)| \le c\,\|u\| \quad \textit{for all} \;\; u \in M,$$

*where $c$ is some positive constant. Then $F$ can be extended to a linear continuous functional $f : X \to \mathbb{R}$ that satisfies*

$$|\langle f, u\rangle| \le c\,\|u\| \quad \textit{for all} \;\; u \in X.$$

First consequences from the Hahn-Banach Theorem are given in the following corollary.

**Corollary 2.7** *Let $X$ be a normed linear space.*

*(i) For each given $u_0 \in X$ with $u_0 \neq 0$, there exists a functional $f \in X^*$ such that*

$$\langle f, u_0\rangle = \|u_0\| \quad \textit{and} \quad \|f\| = 1.$$

*(ii) For all $u \in X$ one has*

$$\|u\| = \sup_{f\in X^*,\ \|f\|\le 1} |\langle f, u\rangle|.$$

*(iii) If for $u \in X$ the condition*

$$\langle f, u\rangle = 0 \quad \textit{for all} \;\; f \in X^*$$

*holds, then $u = 0$.*

We set

$$X^{**} = (X^*)^*,$$