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Advances in Mathematical Logic

Dedicated to the Memory of Professor Gaisi Takeuti, SAML 2018, Kobe, Japan, September 2018, Selected, Revised Contributions

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Preface

Gaisi Takeuti was one of the most brilliant, genius, and influential logicians of the twentieth century. He was born on January 25, 1926, in Ishikawa Prefecture in Japan, graduated from the University of Tokyo in 1947, and received his Ph.D. in mathematical logic from the University of Tokyo in 1956. He was an assistant professor at Tokyo University of Education from 1950, promoted to a professor in 1962, moved to United States in 1966, and had been a professor in mathematics at the University of Illinois at Urbana-Champaign, USA, where he became a professor emeritus after his retirement. He was a member of the Institute for Advanced Study in Princeton during 1959–1960, 1966–1968, and 1971–1972, and the president of the Kurt Gödel Society from 2003 to 2009. He received the Asahi Prize in 1982, and the Bolzano Medal from the Czech Academy of Science in 1996. He passed away on May 10, 2017, at the age of 91.

Takeuti was one of the founders of proof theory, a branch of mathematical logic that originated from Hilbert's program about the consistency of mathematics. Based on Gentzen's pioneering works of proof theory in the 1930s, he proposed a conjecture in 1953 concerning the essential nature of formal proofs of higher-order logic, which is now known as Takeuti's fundamental conjecture. The conjecture is especially important for the foundations of mathematics since a positive solution of the conjecture brings about a finitary proof of the consistency of formalized mathematics, and he gave a partial positive solution of the conjecture by introducing the concept of ordinal diagrams. His arguments on the conjecture and proof theory in general have had great influence on the later developments of mathematical logic, philosophy of mathematics, and applications of mathematical logic to theoretical computer science.

Takeuti's work ranged over the whole spectrum of mathematical logic. In particular, he made significant contributions to set theory, computability theory, Boolean-valued analysis, fuzzy logic, bounded arithmetic, and theoretical computer science. He wrote many monographs and textbooks both in English and in Japanese, and his monumental monograph *Proof Theory*, published in 1975, has long been a standard reference of proof theory. *Introduction to Axiomatic Set Theory*, written with Zaring in 1971, was published as the first volume of Springer Graduate Texts in Mathematics. He had a wide range of interests covering virtually all areas of mathematics and extending to physics. His publications include many Japanese books for students and general readers about mathematical logic, mathematics in general, and connections between mathematics and physics, as well as many essays for Japanese science magazines.

This volume is a collection of papers based on the Symposium on Advances in Mathematical Logic 2018. The symposium had been held from September 18 to 20, 2018, at Kobe University, Japan, and was dedicated to the memory of Professor Gaisi Takeuti. The program and organizing committees are Toshiyasu Arai, Makoto Kikuchi, Satoru Kuroda, Mitsuhiro Okada, and Teruyuki Yorioka. The invited speakers are Samuel R. Buss, Jean-Yves Girard, Kanji Namba, Masanao Ozawa, Wilfried Sieg, and Mariko Yasugi. The contributed speakers are Sakaé Fuchino, Daichi Hayashi, Daisuke Ikegami, Sohei Iwata, Mamoru Kaneko, Takayuki Kihara, Taishi Kurahashi, Hidenori Kurokawa, Yo Matsubara, Tadatoshi Miyamoto, NingNing Peng, Norbert Preining, Toshimichi Usuba, Mariko Yasugi, Keita Yokoyama, and Yasuo Yoshinobu. All the papers contained in this volume are original and refereed.

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Kobe, Japan — Makoto Kikuchi
Gunma, Japan — Satoru Kuroda
Tokyo, Japan — Mitsuhiro Okada
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March 2021 — SAML2018

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Organized by Section of Foundation of Mathematics and History of Mathematical Sciety of Japan

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Symposium on Advances in Mathematical Logic 2018 Dedicated to the Memory of Professor Gaisi Takeuti Program

September 18, 2018

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September 19, 2018

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Reflection Principles, Generic Large Cardinals, and the Continuum Problem

Sakaé Fuchino and André Ottenbreit Maschio Rodrigues

Abstract Strong reflection principles with the reflection cardinal $\leq \aleph_1$ or $< 2^{\aleph_0}$ imply that the size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large. Thus, the stipulation, that a strong reflection principle should hold, seems to support the trichotomy on the possible size of the continuum. In this article, we examine the situation with the reflection principles and related notions of generic large cardinals.

Keywords Continuum problem · Laver-generically large cardinals · Forcing axioms · Reflection principles

1 Gödel's Program and Large Cardinals

The Continuum Problem has been considered to be one of the central problems in set theory. Georg Cantor tried till the end of his mathematical carrier to prove his "theorem" which claims, formulated in present terminology, the continuum, the cardinality $2^{\aleph_0}$ of the set of all real numbers, is the first uncountable cardinal $\aleph_1$. This statement is now called the Continuum Hypothesis (CH). By Gödel [27–29], and Cohen [4–6], it is proven hat CH is independent from the axiom system ZFC of Zermelo-Fraenkel set theory with the Axiom of Choice.[1]

Although the majority of the non-set theorists apparently believes that the results by Gödel and Cohen were the final solutions of the Continuum Problem, Gödel maintained in [30] that the conclusive solution to the problem is yet to be obtained in that a "right" extension of ZFC will be found which will decide the size of the continuum. Today the research program of searching for possible legitimate extensions of ZFC to settle the Continuum Problem is called Gödel's Program. Now

[1]Due to the Incompleteness Theorems, if we would like to formulate this statement[†] precisely, we have to put it under[‡] the assumption that ZFC is consistent (which we not only assume but do believe).

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that, besides CH, a multitude of mathematically significant statements is known to be independent from ZFC, the program should aim to decide not only the size of the continuum but also many of these independent mathematical statements. For modern views on Gödel's Program, the reader may consult e.g. Bagaria [2], Steel [39].

Gödel suggested in [30] that the large cardinal axioms are good candidates of axioms to be added to the axiom system ZFC. Unfortunately large cardinals do not decide the size of the continuum which Gödel also admits in the postscript to [30] added in 1966. Nevertheless, it is known today that some notable structural aspects of the continuum like the Projective Determinacy are decided under the existence of certain large large cardinals.

In this paper, we discuss about a new notion of generic large cardinals introduced in Fuchino, Ottenbreit and Sakai [20] and called there Laver-generic large cardinals (see Sect. 6). Reasonable instances of (the existential statement of a) Laver-generic large cardinal decide the size of the continuum to be either $\aleph_1$ or $\aleph_2$ or fairly large. We show that these three possible scenarios of Laver-generic large cardinal are in accordance with respective strong reflection properties with reflection cardinal $< \aleph_2$ or $< 2^{\aleph_0}$.

In connection with the view-point of set-theoretical multiverse (see Fuchino [16]), our trichotomy theorems, or some further development of them, have certain possibility to become the final answer to the Continuum Problem. As is well-known, Hugh Woodin is creating a theory which should support CH from the point of view of what should hold in a canonical model of the set theory. It should be emphasized that our trichotomy is not directly in contradiction with the possible outcome of his research program. In any case, it should be mathematical results in the future which should decide the matter definitively (if ever?).

2 Reflection Principles

The following type of mathematical reflection properties are considered in many different mathematical contexts.

(2.1) If a structure $\mathfrak{A}$ in the class $\mathcal{C}$ has the property $\mathcal{P}$, then there is a structure $\mathfrak{B}$ in relation $\mathcal{Q}$ to $\mathfrak{A}$ such that $\mathfrak{B}$ has the cardinality $< \kappa$ and $\mathfrak{B}$ also has the property $\mathcal{P}$.

We shall call "$< \kappa$" in (2.1) above the *reflection cardinal* of the reflection property. If κ is a successor cardinal μ^+ we shall also say that the reflection cardinal is $\leq \mu$.

An example of an instance of (2.1) is, when $\mathcal{C}$ = "compact Hausdorff topological spaces", $\mathcal{P}$ = "non-metrizable", $\mathcal{Q}$ = "subspace" and $\kappa = \aleph_2$, that is, with the reflection cardinal $\leq \aleph_1$. In this case, we obtain the statement:

(2.2) For any compact Hausdorff topological space, if X is non-metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable.

This assertion is known to be a theorem in ZFC (see Dow [11]).

If we extend the class $\mathcal{C}$ in (2.2) to $\mathcal{C}$ = "locally compact Hausdorff space", the statement thus obtained

(2.3) For any locally compact Hausdorff topological space, if X is non-metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable

is no more a theorem in ZFC: we can construct a counterexample to (2.3), using a non-reflecting stationary subset S of $E^{\kappa}_{\omega} = \{\alpha < \kappa : cf(\alpha) = \omega\}$ for some regular $\kappa > \omega_1$ (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [18]). Note that $\square_\lambda$ for any uncountable λ implies that there is such S for $\kappa = \lambda^+$. In particular, (2.3) implies the total failure of the square principles and thus we need very large large cardinals to obtain the consistency of this reflection principle. Actually, a known consistency proof of this principle requires the existence of a strongly compact cardinal.[2]

(2.3) is equivalent to the stationarity reflection principle called Fodor-type Reflection Principle (FRP) introduced in [18].[3] This principle can be formulated as follows (see [24]).

For a regular uncountable cardinal λ and $E \subseteq E^{\lambda}_{\omega} = \{\gamma \in \lambda : cf(\gamma) = \omega\}$, a mapping $g : E \to [\lambda]^{\aleph_0}$ is said to be a *ladder system on* E if, for all $\alpha \in E$, $g(\alpha)$ is a cofinal subset of α and $otp(g(\alpha)) = \omega$.

(FRP): *For any regular* $\lambda > \aleph_1$, *stationary* $E \subseteq E^{\lambda}_{\omega}$, *and a ladder system* $g : E \to \lambda^{\aleph_0}$ *on* E, *there is an* $\alpha^* \in E^{\lambda}_{\omega_1}$ *such that*

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

Besides (2.3), there are many mathematical reflection principles in the literature which have been previously studied rather separately but which are now all shown to be equivalent to FRP and hence also equivalent to each other (see [14, 15, 22, 24]). The equivalence of (2.3) to FRP is established in [24] via a further characterization of FRP by non existence of a ladder system with a strong property of disjointness from which a counterexample to (2.3) (and other reflection properties proved to be equivalent to FRP) can be constructed. Here we want to mention only a couple of other reflection statements equivalent to FRP:

For a graph $G = \langle G, \mathcal{E} \rangle$, where $\mathcal{E} \subseteq G^2$ is the adjacency relation of the graph, is said to be *of countable coloring number* if there is a well-ordering $\sqsubset$ on G such that, for each $g \in G$, $\{h \in G : h\ \mathcal{E}\ \text{ and } h \sqsubset g\}$ is finite.

The following assertion is also equivalent to FRP ([18], Fuchino, Sakai Soukup and Usuba [24]):

(2.4) For any graph G, if G is not of countable coloring number, then there is a subgraph H of cardinality $< \aleph_2$ such that H is neither of countable coloring number.

[2] The existence of a strongly compact cardinal is enough to force Rado's Conjecture discussed below and Rado's Conjecture implies the reflection statement (2.3).

[3] Here, we are not only talking about equiconsistency but really about equivalence over ZFC.

In particular, it follows that the assertions (2.3) and (2.4) are equivalent to each other over ZFC.

(Strong) Downward Löwenheim Skolem Theorems of extended logics can be seen also as instances of the scheme (2.1). The following is a theorem in ZFC:

SDLS$(\mathcal{L}(Q), <\aleph_2)$: *For any uncountable first-order structure* $\mathfrak{A}$ *in a countable signature, there is an elementary submodel* $\mathfrak{B}$ of $\mathfrak{A}$ with respect to *the logic* $\mathcal{L}(Q)$ *of cardinality*[4] $< \aleph_2$ *where the quantifier* Q *in a formula "*$Qx\,\varphi$*" is to be interpreted as "there are uncountably many* x such that φ".

Adopting the notation of Fuchino, Ottenbreit and Sakai [19], let $\mathcal{L}^{\aleph_0}_{stat}$ be the logic with monadic (weak) second order variable where the second order variables are to be interpreted as they are running over countable subsets of the structure in consideration. The logic has the built-in predicate ε where atomic formulas of the form $x\varepsilon X$ is allowed for first and second order variables x and X respectively. The logic also has the unique second order quantifier $stat$ which is interpreted by

(2.5) for a structure $\mathfrak{A} = \langle A, ...\rangle$, $\mathfrak{A} \models stat\ X\,\varphi[X, ...]$ holds if and only if $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi[U, ...]\}$ is stationary in $[A]^{\aleph_0}$.

Note that $\mathcal{L}^{\aleph_0}_{stat}$ extends $\mathcal{L}(Q)$ above, since $Qx\,\varphi$ can be expressed by $stat X \exists x\,(x \not\varepsilon X \wedge \varphi)$.

In $\mathcal{L}^{\aleph_0}_{stat}$ we have two natural generalizations of the notion of elementary substructure. For (first order) structures $\mathfrak{A} = \langle A, ...\rangle$ and $\mathfrak{B} = \langle B, ...\rangle$ with $\mathfrak{B} \subseteq \mathfrak{A}$, let

(2.6) $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ if and only if, for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ in the signature of $\mathfrak{A}$, $b_0, ... \in B$, and $U_0, ... \in [B]^{\aleph_0}$, we have $\mathfrak{B} \models \varphi[b_0, ..., U_0, ...] \Leftrightarrow \mathfrak{A} \models \varphi[b_0, ..., U_0, ...]$.

(2.7) $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ if and only if, for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ...)$ in the signature of $\mathfrak{A}$ without any free second order variables, and $b_0, ... \in B$, we have $\mathfrak{B} \models \varphi[b_0, ...] \Leftrightarrow \mathfrak{A} \models \varphi[b_0, ...]$.

By the remark after (2), the following principles are generalizations of SDLS$(\mathcal{L}(Q), <\aleph_2)$:

SDLS$(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$: *For any uncountable first-order structure* $\mathfrak{A}$ *in a countable signature, there is a submodel* $\mathfrak{B}$ *of* $\mathfrak{A}$ *of cardinality* $< \aleph_2$ such that $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

SDLS$^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$: *For any uncountable first-order structure* $\mathfrak{A}$ *in a countable signature, there is a submodel* $\mathfrak{B}$ of $\mathfrak{A}$ *of cardinality* $< \aleph_2$ such that $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

M. Magidor noticed that SDLS$^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies (2.4) (see Magidor [37]). By the equivalence of (2.4) to FRP, we obtain

[4] The cardinality of a structure is defined to be the cardinality of the underlying set.

Theorem 1 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* FRP. □

Actually, it is also easy to see that the stationarity reflection principle RP (which is a strengthening of RP in Jech [31]) follows from $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

FRP follows from our RP ([18]) which is defined as follows:

RP: *For every regular* $\lambda \geq \aleph_2$, *stationary* $S \subseteq [\lambda]^{\aleph_0}$, *and* $X \in [\lambda]^{\aleph_1}$, *there is* $Y \in [\lambda]^{\aleph_1}$ *such that* $cf(Y) = \omega_1$, $X \subseteq Y$ *and* $S \cap [Y]^{\aleph_0}$ *is stationary in* $[Y]^{\aleph_0}$.

Jech's RP is just as our RP as defined above but without demanding the property "$cf(Y) = \omega_1$" for the reflection point Y.

Theorem 2 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* RP.

Sketch of the proof. Let λ, S, X be as in the definition of RP. Let $\mu > \lambda^{\aleph_0}$ be regular and $\mathfrak{A} = \langle \mathcal{H}(\mu), \lambda, S, X, \in\rangle$ where λ, S and X are thought to be interpretations of unary predicate symbols. Let $\mathfrak{B} = \langle B, ...\rangle$ be such that B is of cardinality $\aleph_1$ and $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Then $Y = \lambda \cap B$ is as desired. For example, $cf(Y) = \omega_1$ follows from the fact that $\mathfrak{B} \models \psi$ by elementarity where ψ is the $\mathcal{L}^{\aleph_0}_{stat}$-sentence: $stat\ X\, \exists y (y \underset{\sqcup}{\in} \underset{\sqcup}{\lambda} \wedge \forall z\, ((z\ \varepsilon\ X \wedge z \underset{\sqcup}{\in} \underset{\sqcup}{\lambda}) \rightarrow z \underset{\sqcup}{\in} y))$ where $\underset{\sqcup}{\lambda}$ and $\underset{\sqcup}{\in}$ are constant and binary relation symbols corresponding to λ and $\in$ in the structure $\mathfrak{A}$. □ (Theorem 2)

By a theorem of Todorčević, RP in the sense of Jech implies $2^{\aleph_0} \leq \aleph_2$ (see Theorem 37.18 in [31]). Thus

Corollary 3 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *implies* $2^{\aleph_0} \leq \aleph_2$. □

In contrast to Corollary 3, FRP does not put almost any restriction on the cardinality of the continuum since FRP is preserved by ccc forcing (see [18]).

A proof similar to that of Theorem 2 shows that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the Diagonal Reflection Principle down to an internally club reflection point of cardinality $<\aleph_2$ of S. Cox [8]. Conversely, we can also easily prove that the Diagonal Reflection Principle down to an internally club reflection point of cardinality $<\aleph_2$ implies $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. The internally clubness of the reflection point is used to guarantee that the internal interpretation of the stationary logic coincides with the external correct interpretation of the logic in the small substructure to make it an elementary substructure (in the sense of $\prec^-_{\mathcal{L}^{\aleph_0}_{stat}}$) of the original structure. Thus we obtain (1) of the following theorem.

Theorem 4 (Theorem 1.1, (3) and (4) in [19])

(1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to the Diagonal Reflection Principle down to an internally club reflection point of cardinality* $<\aleph_2$.

(2) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *plus* CH. □

S. Cox proved in [8] that the Diagonal Reflection Principle down to an internally club reflection point of cardinality $\aleph_1$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$. Thus,

Corollary 5 (1) $\mathsf{MA}^{+\omega_1}(\sigma$*-closed*$)$ *implies* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

(2) $\mathsf{MA}^{+\omega_1}(\sigma$*-closed*$) + \mathsf{CH}$ *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. □

The reflection cardinal $<\aleph_2$ (or equivalently $\leq\aleph_1$) in the reflection principles above can be considered to be significant and even natural since, with this reflection cardinal, the reflection principles can be seen as statements claiming that the cardinality $\aleph_1$ is archetypical among uncountable cardinals, and hence that $\aleph_1$ already captures various phenomenon in uncountability in the sense that a certain type of properties of an uncountable structure can be reflected down to a substructure of the cardinality $\aleph_1$. From that point of view, it is interesting that one of the strongest reflection principles, namely the Strong Downward Löwenheim-Skolem Theorem for stationary logic with this reflection cardinal implies CH.

In a similar way, we can also argue that the reflection with the reflection cardinal $<2^{\aleph_0}$ or $\leq 2^{\aleph_0}$ should be regarded as significant and even natural since we can interpret the reflection with these reflection cardinals as a pronouncement of the richness of the continuum.

Let $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ and $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ be the principles obtained from $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ and $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ by replacing "$<\aleph_2$" with "$<2^{\aleph_0}$".

Theorem 6 (Proposition 2.1, Corollaries 2.3, 2.4 in [20])

(1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0}=\aleph_2$. *In particular, if* $2^{\aleph_0}>\aleph_2$, *then* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *does not hold.*

(2) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is inconsistent.* □

Note that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})+\neg\mathsf{CH}$ which is e.g. a consequence of $\mathsf{PFA}^{+\omega_1}$.

Note that Lemma 9 implies that $\mathsf{GRP}^{\omega,\omega_1}(<2^{\aleph_0})$ is also inconsistent.

In contrast to the reflection down to $<2^{\aleph_0}$ whose strong version implies that the continuum is $\aleph_2$ (see Theorem 6, (2) above), the reflection down to $\leq 2^{\aleph_0}$ does not exert any such restriction on the size of the continuum as we will see this in the next section.

A slightly different type of reflection principle with reflection cardinal $<2^{\aleph_0}$ implies that the continuum is very large. We will see this in Sect. 5.

3 Game Reflection Principles and Generically Large Cardinals

There is a further strengthening of $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ which is called (Strong) Game Reflection Principle[5] (GRP) introduced in B. König [34]. The following is a generalization of the principle:

For a regular uncountable cardinal μ, a set A, and $\mathcal{A}\subseteq {}^{\mu>}A$, $\mathcal{G}^{\mu>A}(\mathcal{A})$ is the following game of length μ for players I and II. A match in $\mathcal{G}^{\mu>A}(\mathcal{A})$ looks like:

[5] In [34], B. König originally called the principle introduced here the Strong Game Reflection Principle and the local version of the principle the Game Reflection Principle.

$$\begin{array}{c|cccccccc} I & a_0 & & a_1 & & a_2 & & \cdots a_\xi & & \cdots \\ \hline II & & b_0 & & b_1 & & b_2 \cdots & & b_\xi \cdots \end{array} \qquad (\xi < \mu)$$

where $a_\xi, b_\xi \in A$ for $\xi < \mu$.

II wins this match if

(3.1) $\langle a_\xi, b_\xi : \xi < \eta\rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi : \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A}$ for some $\eta < \mu$; or $\langle a_\xi, b_\xi : \xi < \mu\rangle \in [\mathcal{A}]$

where $\langle a_\xi, b_\xi : \xi < \eta\rangle$ denotes the sequence $f \in {}^{2\cdot\eta}A$ such that $f(2\cdot\xi) = a_\xi$ and $f(2\cdot\xi+1) = b_\xi$ for all $\xi < \eta$ and $[\mathcal{A}] = \{f \in {}^{\mu}A : f \restriction \alpha \in \mathcal{A}\}$ for all $\alpha < \mu$.

For regular cardinals μ, κ with $\mu < \kappa$ $\mathcal{C} \subseteq [A]^{<\kappa}$ is said to be μ-club if $\mathcal{C}$ is cofinal in $[A]^{<\kappa}$ with respect to $\subseteq$ and closed with respect to the union of increasing $\subseteq$-chain of length ν for any regular $\mu \leq \nu < \kappa$.

$\mathsf{GRP}^{<\mu}(<\kappa)$: *For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\mu> A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\mu> B}(\mathcal{A} \cap {}^{\mu>}B)$.*

B. König's *Game Reflection Principle* (GRP) is $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$.

Sometimes, the following variation of the games and the principles is useful: For a limit ordinal δ, a set A, and $\mathcal{A} \subseteq {}^{\delta\geq}A$, $\mathcal{G}^{\delta\geq A}(\mathcal{A})$ is the following game of length δ for players I and II. A match in $\mathcal{G}^{\delta\geq A}(\mathcal{A})$ looks like:

$$\begin{array}{c|cccccccc} I & a_0 & & a_1 & & a_2 & & \cdots a_\xi & & \cdots \\ \hline II & & b_0 & & b_1 & & b_2 \cdots & & b_\xi \cdots \end{array} \qquad (\xi < \delta)$$

where $a_\xi, b_\xi \in A$ for $\xi < \delta$.

II wins this match if

(3.2) $\langle a_\xi, b_\xi : \xi < \eta\rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi : \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A}$ for some $\eta < \delta$; or $\langle a_\xi, b_\xi : \xi < \eta\rangle \in \mathcal{A}$ for all $\eta \leq \delta$.

where $\langle a_\xi, b_\xi : \xi < \eta\rangle$ is defined as above.

For a limit ordinal δ, and uncountable regular cardinals μ, κ with $\delta \leq \mu < \kappa$,

$\mathsf{GRP}^{\delta,\mu}(<\kappa)$: *For any A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\delta\geq A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\delta\geq}A$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\delta\geq B}(\mathcal{A} \cap {}^{\delta\geq}B)$.*

The next Lemma follows immediately from the definitions:

Lemma 7 *Suppose that δ and δ' are limit ordinals and μ, μ', κ, κ' are regular cardinals such that $\delta \leq \delta' < \mu \leq \mu' < \kappa$. Then we have*

(3.3) $\mathsf{GRP}^{<\mu'}(<\kappa) \Rightarrow \mathsf{GRP}^{<\mu}(<\kappa) \Rightarrow \mathsf{GRP}^{\delta',\mu}(<\kappa) \Rightarrow \mathsf{GRP}^{\delta,\mu}(<\kappa)$ □

GRP is indeed a strengthening of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. The following Theorem 8, Lemma 9 and Corollary 10 are slight generalizations of results in B. König [34].

Theorem 8 (Theorem 4.7 in [19]) *Suppose that κ is a regular uncountable cardinal such that*

(3.4) $\mu^{\aleph_0} < \kappa$ *for all* $\mu < \kappa$, *and*
(3.5) $\mathsf{GRP}^{\omega,\omega_1}(<\kappa)$ *holds.*

Then $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *holds.*[6] □

Lemma 9 (Lemma 4.2 in [19]) *For a regular cardinal κ,* $\mathsf{GRP}^{\omega,\omega_1}(<\kappa)$ *implies* $2^{\aleph_0} < \kappa$. □

Remember that GRP is the principle $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$. For a regular cardinal $\kappa > \aleph_1$ we shall write $\mathsf{GRP}(<\kappa)$ for $\mathsf{GRP}^{<\omega_1}(<\kappa)$. Thus GRP is $\mathsf{GRP}(<\aleph_2)$.

Corollary 10 (1) GRP *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.
(2) $\mathsf{GRP}(<(2^{\aleph_0})^+)$ *implies* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0})$.

Proof (1) : By Lemma 9, GRP implies CH. Thus, under GRP, (8) holds for $\kappa = \aleph_2$. By Lemma 7, GRP implies $\mathsf{GRP}^{\omega,\omega_1}(<\aleph_2)$. By Theorem 8, it follows that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

(2) : Note that, for $\mu < (2^{\aleph_0})^+$, $\mu^{\aleph_0} \leq 2^{\aleph_0} < (2^{\aleph_0})^+$ holds. By Lemma 7, $\mathsf{GRP}(<(2^{\aleph_0})^+)$ implies $\mathsf{GRP}^{\omega,\omega_1}(<(2^{\aleph_0})^+)$. Thus, by Theorem 8, it follows that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <(2^{\aleph_0})^+)$, or $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0})$ in the other notation, holds.
□ (Corollary 10)

GRP also implies another prominent reflection principle which is called Rado's Conjecture.

We call a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the initial segment below any element is a well-ordering. A tree $T = \langle T, \leq_T \rangle$ is said to be *special* if it can be partitioned into countably many antichains (i.e. pairwise incomparable sets). Note that every special tree has height $\leq \omega_1$.

For a regular cardinal $\kappa > \aleph_1$, we define Rado's Conjecture with reflection cardinal $<\kappa$ as

$\mathsf{RC}(<\kappa)$: *For any tree T, if T is not special then there is $B \in [T]^{<\kappa}$* such that *B (as the tree $\langle B, \leq_T \cap B^2 \rangle$) is not special.*

The original *Rado's Conjecture* (RC) is $\mathsf{RC}(<\aleph_2)$.

Theorem 11 (B. König [34], see also Theorem 4.3 in [19]) *For a regular cardinal $\kappa > \aleph_1$,* $\mathsf{GRP}^{<\omega_1}(<\kappa)$ *implies* $\mathsf{RC}(<\kappa)$. □

[6] Actually we can prove a slight strengthening of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ (see [19]).

FRP is also a consequence of GRP. This is simply because FRP follows from RC (see [25]).

Game Reflection Principles are characterizations of certain instances of the existence of generically supercompact cardinals.

Let $\mathcal{P}$ be a class of posets. A cardinal κ is said to be a *generically supercompact cardinal by* $\mathcal{P}$, if, for any regular λ, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there are classes M, $j \subseteq \mathsf{V}[\mathbb{G}]$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$, $j : \mathsf{V} \stackrel{\preccurlyeq}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Theorem 12 ([19]) *For a regular uncountable* κ*, the following are equivalent:*

(a) $2^{<\kappa} = \kappa$ *and* $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ *holds.*

(b) κ^+ *is generically supercompact by* $<\kappa$*-closed posets.* □

Corollary 13 (B. König [34]) *The following are equivalent:*

(a) GRP *holds.*

(b) $\aleph_2$ *is generically supercompact by* σ*-closed posets.*

Proof Assume that GRP holds (remember that GRP denotes $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$). Then, by Corollary 10, (1), $2^{<\aleph_1} = 2^{\aleph_0} = \aleph_1$. Thus, by Theorem 12, "(a) $\Rightarrow$ (b)" for $\kappa = \aleph_1$, it follows that $\aleph_2 = (\aleph_1)^+$ is generically supercompact by σ-closed forcing. The implication "(b) $\Rightarrow$ (a)" follows from "(b) $\Rightarrow$ (a)" of Theorem 12 for $\kappa = \aleph_1$. □ (Corollary 13)

4 Simultaneous Reflection down to $< 2^{\aleph_0}$ and $\leq 2^{\aleph_0}$

As we discussed in Sect. 2, the reflection down to $< 2^{\aleph_0}$ as well as the reflection down to $\leq 2^{\aleph_0}$ can be regarded as significant being principles which claim certain richness of the continuum.

One of the strong form of reflection principles with reflection cardinal $< 2^{\aleph_0}$ implies that the continuum is equal to $\aleph_2$ (Theorem 6, (2)) while there is a limitation on the possible types of reflection (Theorem 6, (3)).

In contrast, as we see below, the reflection down to $\leq 2^{\aleph_0}$ can be established in one of its strongest forms without almost any restriction on the size of the continuum: (a) of Theorem 12 can be easily realized starting from a supercompact cardinal.

The following is well-known.

Lemma 14 (Lemma 4.10 in [19]) *If* κ *is a supercompact and* $\mu < \kappa$ *is an uncountable regular cardinal then for* $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ *and* $(\mathsf{V}, \mathbb{P})$*-generic filter* $\mathbb{G}$*, we have* $\mathsf{V}[\mathbb{G}] \models$ *"*$\kappa = \mu^+$ *and* κ *is generically supercompact by* $<\mu$*-closed posets".* □

Suppose now that κ_1 is a supercompact cardinal and $2^{\aleph_0}$ is a regular cardinal. Let $\mathbb{Q} = \mathrm{Col}(2^{\aleph_0}, \kappa_1)$ and let $\mathbb{H}$ be a $(\mathsf{V}, \mathbb{Q})$-generic filter. By $< 2^{\aleph_0}$-closedness of $\mathbb{Q}$, we have $(2^{\aleph_0})^{\mathsf{V}} = (2^{\aleph_0})^{\mathsf{V}[\mathbb{H}]}$ and $\mathsf{V}[\mathbb{H}] \models \kappa_1 = (2^{\aleph_0})^+$. By Lemma 14, $\mathsf{V}[\mathbb{H}] \models$

"$(2^{\aleph_0})^+$is a generically supercompact cardinal by $< 2^{\aleph_0}$-closed posets". By Theorem 12, it follows that $\mathsf{V}[\mathbb{H}] \models$"$\mathsf{GRP}^{<2^{\aleph_0}}(<(2^{\aleph_0})^+)$".

By Corollary 10, (2), Lemma 7 and Theorem 11, we have, in particular,

(4.1) $\mathsf{V}[\mathbb{H}] \models$"$\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, \leq 2^{\aleph_0}) \ \wedge\ \mathsf{RC}(\leq 2^{\aleph_0})$".

Note that the continuum can be forced to be practically anything of uncountable cofinality below κ_1 prior to the generic extension by $\mathbb{Q}$.

The following Proposition 15 should also belong to the folklore (for similar statements, see Theorem 4.1 in König and Yoshinobu [35] or Theorem 4.3 in Larson [36]).

Recall that, for a regular cardinal μ, a poset $\mathbb{P}$ is $<\mu$ *-directed closed* if any downward directed subset of $\mathbb{P}$ of cardinality $<\mu$ has a lower bound (in $\mathbb{P}$).

Proposition 15 *Suppose that* $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ *(or* $\mathsf{PFA}^{+\omega_1}$*, or* $\mathsf{MM}^{+\omega_1}$*, resp.) holds. If* $\mathbb{P}$ *is* $<\aleph_2$*-directed closed, then we have*

(4.2) $\Vdash_{\mathbb{P}}$ "$\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ *(or* $\mathsf{PFA}^{+\omega_1}$*, or* $\mathsf{MM}^{+\omega_1}$*, resp.)*".

Proof We prove the case of $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$. Other cases can be proved by the same argument.

Suppose that $\mathbb{P}$ is a $<\aleph_2$-directed closed poset and let $\underset{\sim}{\mathbb{Q}}$, $\langle \underset{\sim}{D}_\alpha : \alpha < \omega_1\rangle$, $\langle \underset{\sim}{S}_\beta : \beta < \omega_1\rangle$ be $\mathbb{P}$-names such that

(4.3) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is a σ-closed poset,
each $\underset{\sim}{D}_\alpha$ $(\alpha < \omega_1)$ is a dense subset of $\underset{\sim}{\mathbb{Q}}$, and
each $\underset{\sim}{S}_\beta$ $(\beta < \omega_1)$ is a stationary subset of ω_1"

Let $\mathbb{P}^* = \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. For $\alpha < \omega_1$, let

(4.4) $D^*_\alpha = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{P}^* : \mathbb{p} \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{\mathbb{q}} \ \varepsilon\ \underset{\sim}{D}_\alpha\text{"}\}$.

For $\beta < \omega_1$, let

(4.5) $\underset{\sim}{S}^*_\beta = \{\langle\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle, \check{\alpha}\rangle : \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{P}^*,\ \mathbb{p} \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{\mathbb{q}} \Vdash_{\underset{\sim}{\mathbb{Q}}} \text{"}\check{\alpha}\ \varepsilon\ \underset{\sim}{S}_\beta\text{""}\}$.

By the definition of $\mathbb{P}^*$, $\langle D^*_\alpha : \alpha < \omega_1\rangle$, and $\langle \underset{\sim}{S}^*_\beta : \beta < \omega_1\rangle$, the following is easy to show:

Claim 16 $\mathbb{P}^*$ is a σ-closed poset, D^*_α is a dense subset of $\mathbb{P}^*$ for all $\alpha < \omega_1$, and $\underset{\sim}{S}^*_\beta$ is a $\mathbb{P}^*$-name with $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{S}^*_\beta$ is a stationary subset of ω_1" for all $\beta < \omega_1$. □

Let $\mathcal{D}^* = \{D^*_\alpha : \alpha < \omega_1\}$. By $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$, there is a $\mathcal{D}^*$-generic filter $\mathbb{G}^*$ on $\mathbb{P}^*$ such that $\underset{\sim}{S}^*_\beta[\mathbb{G}^*]$ is a stationary subset of ω_1 for all $\beta < \omega_1$.

Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be of cardinality $\aleph_1$ such that $\omega_1 \subseteq M$ and M contains everything relevant (in particular, $\mathbb{G}^* \in M$).

Let $\mathbb{G}_0 = \mathbb{G}^* \cap M$ and let $\mathbb{G}$ be the filter on $\mathbb{P}^*$ generated by $\mathbb{G}_0$. By the choice of M, we have $\underset{\sim}{S}^*_\beta[\mathbb{G}^*] = \underset{\sim}{S}^*_\beta[\mathbb{G}_0] = \underset{\sim}{S}^*_\beta[\mathbb{G}]$.

Let $G = \{\mathbb{p} \in \mathbb{P} : \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}\rangle \in \mathbb{G}$ for some $\underset{\sim}{\mathbb{q}}\}$. Since $|\,G\,| \leq |\,M\,| < \aleph_2$ and G is downward directed, there is a lower bound $\mathbb{p}_0 \in \mathbb{P}$ of G.

Let

(4.6) $\underset{\sim}{\mathbb{H}} = \{\langle \underset{\sim}{q}, \mathbb{1}_{\mathbb{P}} \rangle : \langle p, \underset{\sim}{q} \rangle \in \mathbb{G}$ for some $p \in \mathbb{P}\}$.

Then we have

(4.7) $p_0 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{H}}$ is a $\{\underset{\sim}{D}_\alpha : \alpha < \omega_1\}$-generic filter on $\underset{\sim}{\mathbb{Q}}$ such that
$\underset{\sim}{S}_\beta[\underset{\sim}{\mathbb{H}}]$ is a stationary subset of ω_1 for all $\beta < \omega_1$ ".

Since the argument above can be also performed in $\mathbb{P} \restriction r$ instead of $\mathbb{P}$ for any $r \in \mathbb{P}$. It follows that

(4.8) $\Vdash_{\mathbb{P}}$ "there is a $\{\underset{\sim}{D}_\alpha : \alpha < \omega_1\}$-generic filter H on $\underset{\sim}{\mathbb{Q}}$ such that
$\underset{\sim}{S}_\beta[H]$ is a stationary subset of ω_1 for all $\beta < \omega_1$ ".

□ (Proposition 15)

Theorem 16 *Suppose that κ and κ_1 with $\kappa < \kappa_1$ are two supercompact cardinals. Then there is a generic extension* $\mathsf{V}[\mathbb{G} * \mathbb{H}]$ *such that*

$$\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{MM}^{+\omega_1} + \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0}).$$

Note that, by Corollary 5, (1) , we have

$$\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0}) + \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0}).$$

Proof of Theorem 16 Let $\mathsf{V}[\mathbb{G}]$ be a standard model of MM obtained by a reverse countable iteration of length κ along with a fixed Laver-function $\kappa \to V_\kappa$. It is easy to see that $\mathsf{V}[\mathbb{G}]$ also satisfies $\mathsf{MM}^{+\omega_1}$. Note that we have $\mathsf{V}[\mathbb{G}] \models \kappa = \aleph_2 = 2^{\aleph_0}$. In $\mathsf{V}[\mathbb{G}]$, κ_1 is still supercompact. Thus, working in $\mathsf{V}[\mathbb{G}]$, let $\mathbb{Q} = \mathrm{Col}(2^{\aleph_0}, \kappa_1)$. Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}], \mathbb{Q})$-generic filter. By Proposition 15, we have $\mathsf{V}[\mathbb{G} * \mathbb{H}] \models \mathsf{MM}^{+\omega_1}$. By Lemma 14 and Theorem 12, we have $\mathsf{V}[\mathbb{G} * \mathbb{H}] = (\mathsf{V}[\mathbb{G}])[\mathbb{H}] \models \mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})$.

□ (Theorem 16)

5 Reflection Principles Under Large Continuum

The continuum can be "very large" as a cardinal number. For example, this is the case in the model $\mathsf{V}[\mathbb{G}]$ obtained by starting from a supercompact κ and then adding κ many Cohen reals. In this model, we have $2^{\aleph_0} = \kappa$ and there is a countably saturated normal fine filter over $\mathcal{P}_\kappa(\lambda)$ for all regular $\lambda \geq \kappa$. The last property of $\mathsf{V}[\mathbb{G}]$ implies that κ there is still fairly large (e.g. κ-weakly Mahlo and more, see e.g. Proposition 16.8 in Kanamori [32]).

If the ground model satisfies FRP then $\mathsf{V}[\mathbb{G}]$ also satisfies FRP since FRP is preserved by ccc extensions (see [18]). On the other hand, as we already have seen, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ or even $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ is incompatible with large continuum. In particular, these reflection principles do not hold in our model $\mathsf{V}[\mathbb{G}]$.

A weakening of $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ is compatible with large continuum. Let us begin with the diagonal reflection principle which characterizes the version of the strong downward Löwenheim-Skolem theorem with reflection points of cardinality $<$ large continuum. The following is a weakening of Cox's Diagonal Reflection Principle down to an internally club reflection point.

For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$(*)^{int+}_{<\kappa,\lambda}$: *For any countable expansion* $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$ *and sequence* $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a *is a stationary subset of* $[\mathcal{H}(\lambda)]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda)$, *there are stationarily many* $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(1) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$; *and*
(2) $S_a \cap M$ *is stationary in* $[M]^{\aleph_0}$ *for all* $a \in M$.

Note that (1) implies that $c \subseteq M$ holds for all $c \in [M]^{\aleph_0} \cap M$.

In the notation above, "int" (internal) refers to the condition (2) in which not $S_a \cap [M]^{\aleph_0}$ but $S_a \cap M$ is declared to be stationary in $[M]^{\aleph_0}$; "+" refers to the condition that $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ with (1) and (2) not only exists but there are stationarily many such M.

That $(*)^{int+}_{<\kappa,\lambda}$ is compatible with $\kappa = 2^{\aleph_0}$ and it is arbitrarily large is seen in the following Theorem 17 together with Lemma 18 below:

Theorem 17 (Theorem 2.10 in [20]) *Suppose that* κ *is a generically supercompact cardinal by proper posets. Then* $(*)^{int+}_{<\kappa,\lambda}$ *holds for all regular* $\lambda \geq \kappa$. □

Similarly to Lemma 14, starting from a supercompact cardinal, it is easy to force that the continuum is generically supercompact cardinal by ccc-posets. Let us call a poset $\mathbb{P}$ appropriate for κ, if we have $j''\mathbb{P} \lessdot j(\mathbb{P})$ for all supercompact embedding j for κ.

Lemma 18 *If* κ *is a supercompact and* $\mu < \kappa$ *is an uncountable regular cardinal then for any* $<\mu$*-cc poset* $\mathbb{P}$ *appropriate for* κ*, adding* $\geq \kappa$ *many reals, we have* $\mathsf{V}[\mathbb{G}] \models$ "$\kappa \leq 2^{\aleph_0}$ *and* κ *is generically supercompact by* $<\mu$*-cc posets".* □

"$(*)^{int+}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$" characterizes the strong downward Löwenheim-Skolem theorem for internal interpretation of stationary logic defined in the following.

For a structure $\mathfrak{A} = \langle A, ... \rangle$ of a countable signature, an $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$[7] and $a_0, ... \in A$, $U_0, ... \in [A]^{\aleph_0} \cap A$, we define the internal interpretation of $\varphi(a_0, ..., U_0, ...)$ in $\mathfrak{A}$ (notation: $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$ for "$\varphi(a_0, ..., U_0, ...)$ holds internally in $\mathfrak{A}$") by induction on the construction of φ as follows:

If φ is "$x_i \varepsilon X_j$" then

$$\text{(5.1)} \quad \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...) \;\Leftrightarrow\; a_i \in U_j$$

for a structure $\mathfrak{A} = \langle A, ... \rangle$, $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0} \cap A$.

For first-order connectives and quantifiers in $\mathcal{L}^{\aleph_0}_{stat}$, the semantics "$\models^{int}$" is defined exactly as for the first order "$\models$".

[7] As before, when we write $\varphi = \varphi(x_0, ..., X_0, ...)$, we always assume that the list $x_0, ...$ contains all the free first order variables of φ and $X_0, ...$ all the free weak second order variables of φ.

For an $\mathcal{L}^{\aleph_0}_{stat}$ formula φ with $\varphi = \varphi(x_0, ..., X_0, ..., X)$, assuming that the notion of $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)$ has been defined for all $a_0, ... \in A$, $U_0, ..., U \in [A]^{\aleph_0} \cap A$, we stipulate

(5.2) $\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, ..., U_0, ..., X) \Leftrightarrow$
$\{U \in [A]^{\aleph_0} \cap A : \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\}$ is stationary in $[A]^{\aleph_0}$

for a structure $\mathfrak{A} = \langle A, ...\rangle$ of a relevant signature, $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0} \cap A$.

For structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} = \langle B, ...\rangle$ and $\mathfrak{B} \subseteq \mathfrak{A}$, we define

(5.3) $\mathfrak{B} \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \Leftrightarrow$
$\mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...)$ *if and only if* $\mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$
for all $\mathcal{L}^{\aleph_0}_{stat}$-formulas φ *in the signature of the structures with*
$\varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ *and* $U_0, ... \in [B]^{\aleph_0} \cap B$.

Finally, for a regular $\kappa > \aleph_1$, the internal strong downward Löwenheim-Skolem Theorem $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ is defined by

$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$: *For any structure* $\mathfrak{A} = \langle A, ...\rangle$ *of countable signature with* $|A| \geq \kappa$, *there are stationarily many* $M \in [A]^{<\kappa}$ such that $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

Similarly to the $+$ in "$(*)^{int+}_{<\kappa,\lambda}$", '$+$' in "$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$" refers to the existence of "stationarily many" reflection points M. This additional condition can be dropped if $\kappa = \aleph_2$. This is because the quantifier $Qx\,\varphi$ defined by $stat\, X \exists x\, (x \notin X \wedge \varphi$, $\mathfrak{A} \models^{int} Qx\,\varphi(x, ...))$ still implies that "there are uncountably many $a \in A$ with $\varphi(a, ...)$". Note that, if $\mathfrak{A} \models^{int} \neg stat\, X\, (x \equiv x)$, for a structure $\mathfrak{A} = \langle A, ...\rangle$, we can easily find even club many $X \in [A]^{<\kappa}$ for any regular $\aleph_1 \leq \kappa \leq |A|$ such that $\mathfrak{A} \restriction X \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

Proposition 19 (Proposition 3.1 in [20]) *For a regular cardinal* $\kappa > \aleph_1$*, the following are equivalent:*

(a) $(*)^{int+}_{<\kappa,\lambda}$ *holds for all regular* $\lambda \geq \kappa$.

(b) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *holds.* □

Although $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ is compatible with large continuum, as a weakening of $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$, this principle does not imply the largeness of the continuum. The strong Löwenheim-Skolem theorem for the following variation of stationary logic does.

For sets s and t we denote with $\mathcal{P}_s(t)$ the set $[t]^{<|s|} = \{a \in \mathcal{P}(t) : |a| < |s|\}$. We say $S \subseteq \mathcal{P}_s(t)$ is stationary if it is stationary in the sense of Jech [31].

The logic $\mathcal{L}^{\mathsf{PKL}}_{stat}$ has a built-in unary predicate symbol $\underline{K}(\cdot)$.[8] For a structure $\mathfrak{A} = \langle A, \underline{K}^{\mathfrak{A}}, ...\rangle$, the weak second-order variables X, Y,... run over elements of $\mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A)$.

[8] PKL stands here for "pi-kappa-lambda" in the sense of "$\mathcal{P}_\kappa(\lambda)$".