Xiao-Jun Yang

Theory and Applications of Special Functions for Scientists and Engineers

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To my family, parents, brother, sister, wife,
and my daughters

Preface

The main target of this monograph is to provide the detailed investigations to the newly established special functions involving the Mittag-Leffler, Wiman, Prabhakar, Miller–Ross, Rabotnov, Lorenzo–Hartley, Sonine, Wright, and Kohlrausch–Williams–Watts functions, Gauss hypergeometric series, and Clausen hypergeometric series. The integral transform operators based on the theory of the Wright and Kohlrausch–Williams–Watts functions may be used to solve the complex problems with power-law behaviors in the light of nature complexity. The topics are important and interesting for scientists and engineers to represent the complex phenomena arising in mathematical physics, engineering, and other applied sciences.

The monograph is divided into seven chapters, which are discussed as follows.

Chapter 1 introduces the special functions such as Euler gamma function, Pochhammer symbols, Euler beta function, extended Euler gamma function, extended Euler beta function, Gauss hypergeometric series, and Clausen hypergeometric series as well as calculus operators with respect to monotone function containing the power-law calculus, scaling-law calculus, and complex topology calculus as well as calculus operators with respect to logarithmic and exponential functions.

Chapter 2 investigates the Wright function, Wright's generalized hypergeometric function, supertrigonometric and superhyperbolic functions via Wright function, and Wright's generalized hypergeometric function. The integral representations for the supertrigonometric and superhyperbolic functions are addressed in detail. Some integral transforms via Dunkl transform based on the calculus with respect to power-law function are proposed.

Chapter 3 provides the theory of the Mittag-Leffler function, supertrigonometric functions, and superhyperbolic functions. The integral representations for the Mittag-Leffler function and related functions are addressed, and the general fractional calculus operators are also discussed in detail. The truncated Mittag-Leffler, supertrigonometric, and superhyperbolic functions are considered, and some mathematical models are considered to explain the power-law behaviors in material science.

Chapter 4 shows the theory of the Wiman function, supertrigonometric functions, and superhyperbolic functions. The integral representations for the Wiman function and related functions are addressed, and the general fractional calculus operators are also discussed in detail. The truncated Wiman, supertrigonometric, and superhyperbolic functions are considered, and the integral equations as well as mathematical models related to Wiman function are also presented in detail.

Chapter 5 addresses the theory of Prabhakar function and proposes the supertrigonometric and superhyperbolic functions via Prabhakar function. The Laplace transforms for the new special functions and integral representations for the supertrigonometric and superhyperbolic functions are discussed in detail. The truncated Prabhakar, supertrigonometric, and superhyperbolic functions are proposed, and the general fractional calculus involving the Prabhakar function is considered. The integral equations and mathematical models related to Prabhakar function are also presented.

Chapter 6 presents the Sonine functions, Rabotnov fractional exponential function, Miller–Ross function, and Lorenzo–Hartley functions. The Laplace and Mellin transforms of them are given, and the integral representations for the supertrigonometric and superhyperbolic functions are also presented in detail. The formulas related to the Mittag-Leffler functions and Wright hypergeometric functions are also considered.

Chapter 7 illustrates the Kohlrausch–Williams–Watts function and integral representations. The subtrigonometric functions, subhyperbolic functions, supertrigonometric functions, and superhyperbolic functions are discussed in detail. Moreover, the Fourier-type series, Fourier-type integral transforms, Laplace-type integral transforms, and Mellin-type integral transforms are also proposed.

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Xiao-Jun Yang

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Contents

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Xiao-Jun Yang, PhD, is Professor of Applied Mathematics and Mechanics at China University of Mining and Technology, Xuzhou, China. His scientific interests include mathematical physics, fractional calculus and applications, fractals, mechanics, analytic number theory, integral transforms, and special functions. He is a recipient of the Atanasije Stojkovič Medal, Belgrade, Serbia (2021). Professor Yang was awarded the Abel Award (Istanbul, Turkey, 2020) for his achievements in the area of fractional calculus and its applications. He was also awarded the Obada-Prize, Cairo, Egypt (2019). He is a recipient of the Young Scientist Award (2019) for contributions in developing local fractional calculus at ICCMAS-2019, Istanbul, Turkey, and the Springer Distinguished Researcher Award (2019) at ICMMAAC-2019, Jaipur, India. He is a highly cited researcher (2021, 2020, and 2019, Clarivate Analytics) in mathematics and Elsevier Most Cited Chinese Researcher in Mathematics (2017, 2018, 2019, and 2020). Professor Yang is one of the scientific committee members of 10th edition of the Pan African Congress of Mathematicians. He is the author and co-author of seven monographs for Elsevier, Springer Nature, CRC, World Science, and Asian Academic, and co-editor of one edited book for De Gruyter.

Chapter 1
Preliminaries

Abstract In this chapter, we investigate the special functions and operator calculus. At first, the Euler gamma function, Pochhammer symbols, Euler beta function, extended Euler gamma function, and extended Euler beta function are introduced. Then, the Gauss hypergeometric series, Clausen hypergeometric series, super-trigonometric and superhyperbolic functions, and Laplace and Mellin transforms are presented. Finally, the calculus operators with respect to monotone function are discussed and the mathematical models in applied sciences are also reported in detail.

1.1 The Euler Gamma Function, Pochhammer Symbols, Euler Beta Function, and Related Functions

In this section, we present the Euler gamma function, Pochhammer symbols, Euler beta function, extended Euler gamma function, and extended Euler beta function.

1.1.1 The Euler Gamma Function

In this part, we introduce the Euler gamma function.

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ be the sets of the complex numbers, real numbers, integrals, and natural numbers, respectively.

Let $\mathbb{Z}^+$, $\mathbb{R}_+$, $\mathbb{Z}^-$, and $\mathbb{R}_-$ be the sets of the positive integrals, positive real numbers, and negative integral numbers, and negative real numbers.

Let $\mathbb{Z}_0^- = \mathbb{Z}^- \cup 0$ and $\mathbb{N}_0 = \mathbb{N} \cup 0$.

Let $Re\,(x)$ denote the real part of x if $x \in \mathbb{C}$.

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Definition 1.1 (Euler [1]) The gamma function due to Euler is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.1}$$

where $Re(z) > 0$ and $z \in \mathbb{C}$.

The formula was discovered by Euler in 1729 (see [1], p.1), and the notation $\Gamma(z)$ was introduced by Legendre in 1814 (see [2], p.476).

Theorem 1.1 (Weierstrassian Product [3]) *If* $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *with* $\mathbb{Z}_0^- =: \{0, -1, -2, \cdots\}$ *and* $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right)$ *is the Euler constant, the Gamma function was given as [3]*

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right)^{-1} e^{\frac{z}{k}}\right). \tag{1.2}$$

Moreover, $\Gamma(z)$ *is analytic except at the points* $z \in \mathbb{Z}_0^-$*, where it has simple poles [4].*

The formula for the Weierstrassian product was discovered by Weierstrass in 1856 [3] and by Newman in 1848 [5], respectively, and the proofs were published by Hölder [6], Moore [7], and Baines [8].

Definition 1.2 (Euler [1]) Let $Re(z) > 0$ and $z \in \mathbb{C}$. Then the Euler's functional equation states

$$\Gamma(z+1) = z\Gamma(z). \tag{1.3}$$

The result is the Euler's functional equation discovered by Euler in 1729 [9] and reported by Weierstrass [3], Brunel [10], Gronwall [11], and Olver [12].

Theorem 1.2 (Euler [1]) *If* $z \in \mathbb{N}_0$*, then we have*

$$\Gamma(z+1) = z!. \tag{1.4}$$

The result is the Euler's functional equation discovered by Euler in 1729 [1, 2] and discussed by Weierstrass [4], Brunel [10], and Gronwall [11].

Theorem 1.3 (Euler [1])

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1.5}$$

This work was discovered by Euler in 1729 [11] and discussed in Bell [13], Luke [14], and Bendersky [15].

Theorem 1.4 (Euler) *If* $n, j \in \mathbb{N}$, *then we have*

$$\prod_{j=1}^{n} \Gamma\left(\frac{1-j}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}}. \tag{1.6}$$

The result was reviewed by Gronwall in 1916 [11].

Theorem 1.5 (Winckler [16]) *If* $z \in \mathbb{C}$ *and* $g, k, j, k, l, m, n \in \mathbb{N}$, *then we have*

$$\frac{\prod_{j=0}^{n-1} \Gamma\left(hz + \frac{hj}{n}\right)}{\prod_{l=0}^{m-1} \Gamma\left(gz + \frac{gl}{m}\right)} = \left(\frac{h}{g}\right)^{hgz + \frac{hg-h-g}{2}} (2\pi)^{\frac{h-g}{2}}. \tag{1.7}$$

The result was discovered by Winckler in 1856 [16] and reviewed by Gronwall in 1916 [11].

Theorem 1.6 (Schlömilch [17] and Newman [5]) *If* $z \in \mathbb{C}$ *and* $k \in \mathbb{N}$, *then we have*

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}. \tag{1.8}$$

The result was discovered by Schlömilch in 1844 [17] and by Newman [18].

Theorem 1.7 (Whittaker [19]) *If* $Re(z) > 0$, $z \in \mathbb{C}$, *and* $k \in \mathbb{N}$, *then we have*

$$\int_{0}^{\infty} e^{-kt} t^{z-1} dt = \frac{\Gamma(z)}{k^z}. \tag{1.9}$$

The result was first reported by Whittaker in 1902 (see [19], p.184) and further reported by Whittaker and Watson in 1920 [20].

Theorem 1.8 (Whittaker [19]) *If* $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$, *and* $Re(\beta) > 0$, *then we have*

$$\int_{0}^{\frac{\pi}{2}} \cos^{\alpha-1} t \sin^{\beta-1} t dt = \frac{1}{2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+\beta}{2}\right)}. \tag{1.10}$$

The result was first defined by Whittaker in 1902 (see [19], p.191) and further reported by Whittaker and Watson in 1920 [20].

Theorem 1.9 (Titchmarsh [21]) *If $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$, and $Re(\alpha + \beta) > 1$, then we have*

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha + t)\,\Gamma(\beta - t)} dt = \frac{2^{\alpha+\beta-1}}{\Gamma(\alpha + \beta - 1)}. \tag{1.11}$$

The result was first reported in the Titchmarsh's monograph [21].

Theorem 1.10 (Titchmarsh [21]) *If $Re(\alpha) > -1$, $Re(\beta) > -1$, and $Re(\alpha + \beta) > -1$, then we have*

$$f \prod_{k=1}^{\infty} \frac{k(\alpha + \beta + k)}{(\alpha + k)(\beta + k)} = \frac{\Gamma(\alpha + 1)\,\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)}. \tag{1.12}$$

The result was first presented in the Titchmarsh's monograph [21].

Theorem 1.11 (Titchmarsh [21]) *If $z \in \mathbb{C}$ and $k, n \in \mathbb{N}$, then we have*

$$\prod_{k=1}^{n} \left(1 - \frac{z}{k^n}\right) = \left(-\prod_{k=1}^{n} \Gamma\left(-e^{2\pi i \frac{k-1}{n}} z^{\frac{1}{n}}\right)\right)^{-1}. \tag{1.13}$$

The result was first reported in the Titchmarsh's monograph [21].

Theorem 1.12 (Euler [22]) *Let $z \in \mathbb{C}$ and $Re(z) > 0$. Then we have the Euler's completion formula as follows:*

$$\Gamma(z)\,\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \tag{1.14}$$

and

$$\sin(\pi z) = \pi z \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right). \tag{1.15}$$

The result is the Euler's completion formula due to Euler [22].

For more details of the results, readers refer to Weierstrass [4], Manocha and Srivastava [23], Luke [14], Bell [13], Godefroy [24] and Tannery [25].

Theorem 1.13 (Legendre [2], p.485)

*The **Legendre duplication formula** states*

$$\Gamma(2z)\,\Gamma\left(\frac{1}{2}\right) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right), \tag{1.16}$$

where $z \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

The Legendre's duplication formula was first discovered by Legendre in 1809 (see [2], p.477). For more details of the Legendre duplication formula, readers refer to Gronwall [11], Andrews et al. [26], and Manocha and Srivastava [23].

Theorem 1.14 (Gauss [27]) *If* $z \in \mathbb{C} \backslash \left\{0, -\frac{j}{m}\right\}$ *with* $j < m$ *and* $j, m \in \mathbb{N}$*, then we have*

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left(z + \frac{j-1}{m}\right). \tag{1.17}$$

The result is the Gauss' multiplication formula due to Gauss [27]. For more details of the Gauss' multiplication formula, readers refer to Winckler [28], Gronwall [11], Manocha and Srivastava [23], and Andrews et al. [26].

Theorem 1.15 (Weierstrass [3]) *If* $z \in \mathbb{C}/\mathbb{Z}_0^-$*, then we have*

$$\begin{aligned} &\Gamma\left(\tfrac{1}{2} - z\right)\Gamma\left(\tfrac{1}{2}+z\right) \\ &= \pi sec\,(\pi z) \\ &= \tfrac{\pi}{\cos(\pi z)} \\ &= \tfrac{2\pi}{e^{i\pi z}+e^{-i\pi z}}. \end{aligned} \tag{1.18}$$

The result was discovered by Weierstrass [4] and reported by Bell in 1968 [13] and by Luke in 1969 [14].

Theorem 1.16 $\Gamma\left(\frac{1}{2} + iz\right)\Gamma\left(\frac{1}{2} - iz\right) = \frac{\pi}{\cosh(\pi z)} = \frac{2\pi}{e^{\pi z}+e^{-\pi z}}$, *(19)*

$$\left|\Gamma\left(\frac{1}{2} + iz\right)\right|^2 = \frac{2\pi}{e^{\pi z} + e^{-\pi z}} \tag{1.19}$$

and

$$\Gamma(iz)\,\Gamma(-iz) = \frac{\pi}{-iz\sin(\pi z i)} = \frac{2\pi}{z\left(e^{\pi z} - e^{-\pi z}\right)}, \tag{1.20}$$

where $|z| \to \infty$.

The results were reported by different researchers, for example, Lerch [29], Godefroy [30], Stieltjes [31], Bateman [32], and Andrews et al. [26].

Theorem 1.17

$$\prod_{j=1}^{n-1} \Gamma\left(\frac{j}{n}\right)\Gamma\left(1 - \frac{j}{n}\right) = \frac{(2\pi)^{n-1}}{n}, \tag{1.21}$$

$$\Gamma\left(-n+\frac{1}{2}\right)=(-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!},\tag{1.22}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!\sqrt{\pi}}{2^n},\tag{1.23}$$

$$\Gamma(n+z)\,\Gamma(n-z)=\frac{\pi z}{\sin(\pi z)}\left((n-1)!\right)^2\prod_{j=1}^{n-1}\Gamma\left(1-\frac{z^2}{j}\right)\tag{1.24}$$

and

$$\Gamma\left(n+\frac{1}{2}+z\right)\Gamma\left(n+\frac{1}{2}-z\right)=\frac{\left(\Gamma\left(n+\frac{1}{2}\right)\right)^2}{\cos(\pi z)}\prod_{j=1}^{n}\Gamma\left(1-\frac{4z^2}{(2j-1)^2}\right),\tag{1.25}$$

where $n\in\mathbb{N}$ *and* $z\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

The results were reported by Weierstrass in 1856 [4].

Let us introduce the Temme function which is related to the ratio of two gamma functions [35].

Definition 1.3 The Temme function is defined as

$$\Gamma^*(z)=\frac{\Gamma(z)}{\sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z}},\tag{1.26}$$

where $z\in\mathbb{C}$ and $Re(z)>0$.

The result was defined by in Temme's book (see [35], p.66).

Theorem 1.18 *Let* $z,a,b\in\mathbb{C}$, $Re(z)>0$, $Re(z+a)>0$, *and* $Re(z+b)>0$.

Then the ratio of two gamma functions is shown as follows:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{\alpha-b}\frac{\Gamma^*(z+a)}{\Gamma^*(z+b)}Q(z,a,b),\tag{1.27}$$

where

$$Q(z,a,b)=\left(1+\frac{a}{z}\right)^{a-\frac{1}{2}}\left(1+\frac{b}{z}\right)^{\frac{1}{2}-b}e^{z\left[\ln\left(1+\frac{a}{z}\right)-\frac{a}{z}-\ln\left(1+\frac{b}{z}\right)+\frac{b}{z}\right]}.\tag{1.28}$$

The result was discovered by Temme in the book (see [35], p.67).

In an alternative manner, we have (see [35], p.67)

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{1}{\Gamma(b-a)} \int_0^1 t^{z+a-1} (1-t)^{b-a-1} \, dt, \tag{1.29}$$

where $a, b, z \in \mathbb{C}$ and $Re\,(b-a) > 0$.

The result was reported by Temme in the book (see [35], p.67).

Here, we introduce the interested formula reported in the book (see [35], p.72) as follows:

Theorem 1.19 *Let $a, b, z \in \mathbb{C}$, $Re\,(a) > 0$, $Re\,(b) > 0$, and $Re\,(z) > 0$.*

Then we have

$$\int_0^\infty t^{z-1} e^{-at^b} dt = \frac{1}{b} \Gamma\left(\frac{z}{b}\right) a^{-\frac{z}{b}}. \tag{1.30}$$

The result was reported by Temme (see [35], p.72).

There are some special cases of (1.29) as follows:

$$\int_0^\infty t^{z-1} e^{-at} dt = a^{-z} \Gamma(z), \tag{1.31}$$

$$\int_0^\infty e^{-t^b} dt = \Gamma\left(\frac{1}{b} + 1\right), \tag{1.32}$$

$$\int_0^\infty e^{-at^b} dt = \frac{1}{b} \Gamma\left(\frac{z}{b}\right) a^{-\frac{z}{b}}, \tag{1.33}$$

$$\int_0^\infty e^{-at^2} dt = \frac{1}{2} \Gamma\left(\frac{z}{2}\right) a^{-\frac{z}{2}}, \tag{1.34}$$

$$\int_0^\infty t^{z-1} e^{-at^2} dt = \frac{1}{2} \Gamma\left(\frac{z}{2}\right) a^{-\frac{z}{2}}, \tag{1.35}$$

$$\int_0^\infty t^{z-1} e^{-t^2} dt = \frac{1}{2} \Gamma\left(\frac{z}{2}\right) \tag{1.36}$$

and

$$\int_0^\infty t^{z-1} e^{-t^b} dt = \frac{1}{b}\Gamma\left(\frac{z}{b}\right), \tag{1.37}$$

where $a, b, z \in \mathbb{C}$, $Re\,(a) > 0$, $Re\,(b) > 0$, and $Re\,(z) > 0$.

There are useful formulas as follows [35]:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}, \tag{1.38}$$

$$\Gamma\left(-n+\frac{1}{2}\right) = (-1)^n \sqrt{\pi} 2^{2n} \frac{n!}{(2n)!}, \tag{1.39}$$

$$\Gamma\left(\frac{1}{2} - z\right)\Gamma\left(z+\frac{1}{2}\right) = \frac{\pi}{\cos(\pi z)}, z - \frac{1}{2} \notin \mathbb{Z}, \tag{1.40}$$

$$\Gamma\left(\frac{1}{2} - xi\right)\Gamma\left(xi+\frac{1}{2}\right) = \frac{\pi}{\cosh(\pi x)}, \tag{1.41}$$

$$\int_0^\infty t^{z-1} \sin t dt = \Gamma(z) \sin\frac{\pi z}{2} \tag{1.42}$$

and

$$\int_0^\infty t^{z-1} \cos t dt = \Gamma(z) \cos\frac{\pi z}{2}, \tag{1.43}$$

where $z \in \mathbb{C}$, $Re\,(z) > 0$, $n \in \mathbb{N}$ and $x \geq 0$.

Making use of (1.30) and taking $a = i = \sqrt{-1}$, we have

$$\int_0^\infty t^{z-1} e^{-it} dt = i^{-z}\Gamma(z), \tag{1.44}$$

$$\begin{aligned} \int_0^\infty t^{z-1} e^{-it} dt &= \int_0^\infty t^{z-1} (\cos t - i \sin t)\, dt \\ &= \int_0^\infty t^{z-1} \cos t dt - i \int_0^\infty t^{z-1} \sin t dt \end{aligned} \tag{1.45}$$

and

$$\begin{aligned} i^{-z}\Gamma(z) &= e^{-\frac{z\pi i}{2}}\Gamma(z) \\ &= \left(\cos\tfrac{z\pi}{2} - \sin\tfrac{z\pi}{2}\right)\Gamma(z), \end{aligned} \tag{1.46}$$

where $(i\theta)^z = |\theta|\, e^{-\frac{z\pi i sgn(z)}{2}}$, $z \in \mathbb{C}$ and $Re(z) > 0$, such that we obtain (1.42) and (1.43).

1.1.2 The Pochhammer Symbols and Related Formulas

We now introduce the Pochhammer symbols and related theorems.

Definition 1.4 (Pochhammer [34])
The Pochhammer symbol is defined as [34]

$$\begin{aligned} (\alpha)_k &= \prod_{n=1}^{k}(\alpha+n-1) \\ &= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \\ &= \begin{cases} 1 \; (k=0) \\ \alpha(\alpha+1)\cdots(\alpha+k-1) \; (k \in \mathbb{N}_0) \end{cases} \end{aligned} \tag{1.47}$$

and

$$(\alpha)_0 = 1, \tag{1.48}$$

where $\alpha \in \mathbb{C}$ and $k, n \in \mathbb{N}$.

The Pochhammer symbol was first suggested by Pochhammer in 1870 [34].
The notation was first used by Pochhammer in 1870 [34] and Weierstrass noticed in 1856 that [4]

$$\Gamma(\alpha+k) = \alpha(\alpha+1)\cdots(\alpha+k-1)\,\Gamma(\alpha) \; (k \in \mathbb{N}_0). \tag{1.49}$$

For more information, readers may refer to the monograph [33].
Moreover, there is (see[4, 11, 36])

$$\lim_{k\to\infty}(\alpha)_k = \frac{1}{\Gamma(\alpha)}, \tag{1.50}$$

where $\alpha \in \mathbb{C}\backslash\mathbb{C}_0^-$ and $k \in \mathbb{N}$.

Suppose that $\alpha = -n$ and $n \in \mathbb{N}_0$, then there is (see [36], p.3)

$$(\alpha)_k = \begin{cases} (-n)_k \,, n \geq k \\ -, n < k. \end{cases} \tag{1.51}$$

Theorem 1.20 (Euler [1]) *If $z \in \mathbb{C}\backslash\mathbb{Z}_0^-$, then we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z}{(z)_{n+1}}. \tag{1.52}$$

The result was discovered by Euler in 1729 [9], reported by Weierstrass in 1856 [3], and discussed by Gronwall in 1916 [11].

Theorem 1.21 *There exist*

$$(\alpha)_k \, (\alpha + k)_n = (\alpha)_{n+k} \,, \tag{1.53}$$

and

$$(\alpha + k)_{m-k} = \frac{(\alpha)_m}{(\alpha)_k}, \tag{1.54}$$

where $\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $n, k \in \mathbb{N}$.

The first formula of the results was reported by Rainville in 1960 (see [37], p.59) and the second formula was suggested by Slater in 1966 (see [36], p.31).

There are some useful formulas as follows:

$$(-z)_n = (-1)^n \, (z - n + 1)_n, \tag{1.55}$$

$$(z)_{2n} = 2^{2n} \left(\frac{z}{2}\right)_n \left(\frac{z}{2} + \frac{1}{2}\right)_n, \tag{1.56}$$

$$(z)_{2n+1} = 2^{2n+1} \left(\frac{z}{2}\right)_{n+1} \left(\frac{z}{2} + \frac{1}{2}\right)_n, \tag{1.57}$$

where $z \in \mathbb{C}$ and $n \in \mathbb{N}$.

Theorem 1.22 *Let $j, k, m, n \in \mathbb{N}_0$, $k \leq n$ and $\alpha, \beta \in \mathbb{C}\backslash\mathbb{Z}$, then we have*

$$(1)_n = n!, \tag{1.58}$$

$$\begin{pmatrix}\alpha\\ n\end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-k-1)}{n!} \\ = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} \\ = \frac{1}{n!(\alpha+1)_n} \\ = \frac{(-1)^n(-\alpha)_n}{n!}, \tag{1.59}$$

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} = (-1)^n(-\alpha)_n\,, \tag{1.60}$$

$$\frac{1}{(m-n)!} = \frac{(-1)^n(-m)_n}{m!}, \tag{1.61}$$

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}, \tag{1.62}$$

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = (\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}, \tag{1.63}$$

$$\frac{n!}{(\alpha)_{n+1}} - \frac{n!}{(\alpha+1)_{n+1}} = \frac{(n+1)!}{(\alpha)_{n+2}}, \tag{1.64}$$

$$\frac{(\alpha)_n}{(\beta)_n} - \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} = \frac{(\alpha)_n}{(\beta)_{n+1}}(\alpha-\beta)\,, \tag{1.65}$$

$$(\alpha)_{n-k} = \frac{(-1)^k}{(1-\alpha)_{-n}(1-\alpha-n)_k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}, \tag{1.66}$$

$$(1)_{n-k} = (n-k)! = \frac{(-1)^k(1)_n}{(-n)_k} = = \frac{(-1)^k n!}{(-n)_k}, \tag{1.67}$$

$$(\alpha)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\alpha+j-1}{m}\right)_n, \tag{1.68}$$

and

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!}, (0 \le k \le n) \\ 0, (k > n) \end{cases}. \tag{1.69}$$

For more details of the results, readers refer to the works [3, 11, 13, 14, 23, 36].

Theorem 1.23 (Stirling [38])

$$\begin{aligned} \frac{1}{z-\alpha} &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(z)_{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(z)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\Gamma(z+k+1)} \\ &= \frac{1}{z} + \frac{\alpha}{z(z+1)} + \frac{\alpha(\alpha+1)}{z(z+1)(z+2)} + \cdots +, \end{aligned} \tag{1.70}$$

where $Re(\alpha) > 0$*,* $Re(z) > 0$*,* $Re(\alpha - z) > 0$ *and* $\alpha, z \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The result was discovered by Stirling in 1730 [38] and reviewed by Gronwall in 1916 [11].

Theorem 1.24 *Let* $h^{(1)}(t) > 0, h(0) = a, h(1) = b$*,* $x \in \mathbb{C}$ *and* $Re(x) > 0$*.*
Then we have

$$\Gamma(x) = \int_a^b e^{-h(t)} (h(t))^{x-1} h^{(1)}(t)\, dt\,. \tag{1.71}$$

The result was discovered by Yang *et al.* in 2020 when $x \in \mathbb{N}$ [39].

1.1.3 The Euler Beta Function

In this section, we investigate the concept and theorems of the Euler beta function.

Definition 1.5 (Euler [22])
The Euler beta function is defined as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt, \tag{1.72}$$

where $Re(\alpha) > 0$, $Re(\beta) > 0$, and $\alpha, \beta \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

The formula (called the Euler integral of the second kind) was first discovered by Euler in 1772 [22] and by Legendre in 1811 (see [40], p.211), and the name of the beta function was introduced for the first time by Binet in 1839 [41]. For more details, see the monograph [33].

It is clear that

$$B(\alpha, 1) = \frac{1}{\alpha+1} \tag{1.73}$$